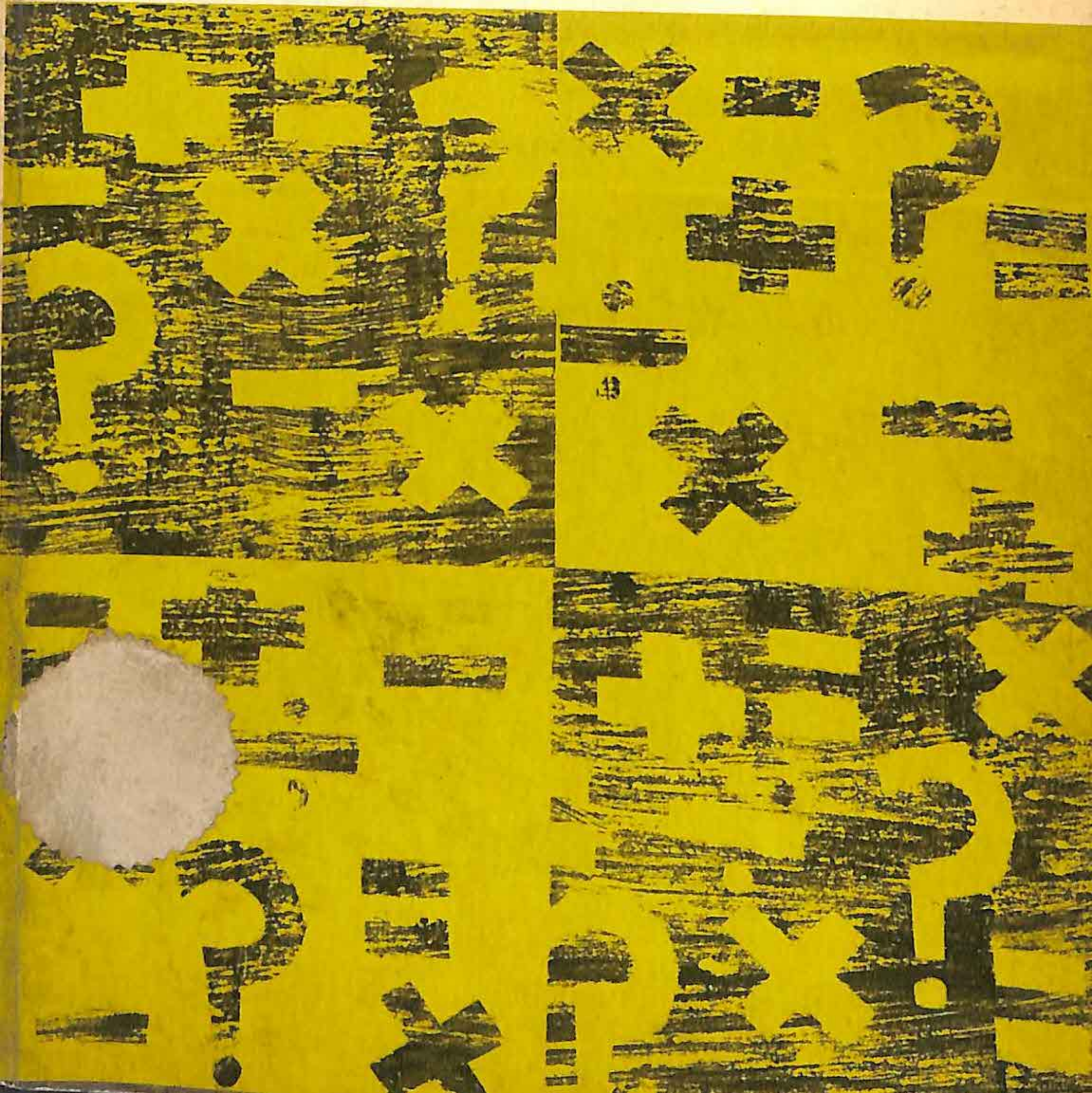
Evaluating Pupils' Understanding of Arithmetic

WILBUR H. DUTTON



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Preface

During the past two decades considerable emphasis has been placed upon the meaningful teaching of arithmetic. This approach to teaching received its original impetus from psychologists who were proving the advantages of learning with understanding as opposed to learning by isolated drill. Much progress has been made in identifying the concepts, principles, relationships, and generalizations essential to the meaningful learning of airthmetic. In spite of these gains, there has been a wide gap between what is known about learning theory and what is being applied in the teaching of arithmetic.

The revolution now taking place in elementary school mathematics has progressed so rapidly that there is a real danger that the meaningful approach to teaching may be disregarded or greatly altered. The process of meshing the old with the new becomes an important problem to all who work with or who are concerned about the teaching of arithmetic.

This book has as its major purpose the identification of valid techniques to evaluate meaningful teaching and learning of both traditional and modern elementary school mathematics programs. For those individuals who do not understand the meaningful approach to teaching, a brief historical treatment and review of research studies supporting this approach have been provided.

A major difficulty in the evaluation of meaningful teaching in the past has been the dearth of suitable tests. The author has prepared tests to measure the major arithmetic understandings taught in grades three through six. These tests have been applied in several typical school districts and the results used to show various methods of interpretation of data. Practical applications of evaluation procedures in the classroom have been given to encourage teachers to begin comprehensive evaluation work. If the teacher tries new ways to evaluate pupil understanding of the new mathematical concepts introduced into the elementary school curriculum, there is a good chance that the teacher will keep pace with these modern developments in mathematics.

A number of graduate students and colleagues at the University of California have influenced the preparation of this book. My appreciation is expressed to all of those students in graduate seminars who helped criticize test items and who assisted in the administration of tests in the school districts reported on in Chapter IV. To my office staff much credit is due for typing and handling of numerous details involved in working with the whole project. I am especially thankful to my wife, who has been proofreader, typist, critic, and sympathetic advisor during the preparation of the entire manuscript.

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I

Modern Programs in Elementary School Mathematics

No other educational problem is receiving as much attention as the new mathematics curricula. We are, in fact, in the midst of a revolution which began after World War II and which is assuming amazing proportions today. Persistent problems relating to the *kind* and amount of mathematics to be taught in the elementary school have been accentuated. Problems pertaining to the teaching of arithmetic have been created by the introduction of new mathematical content and recent interpretations of the meaning theory. Because of rapid developments within the field of mathematics and the rapidity of change within our culture, it is not clear what the content or methods will be a few years from now. Thus the need for pupils and teachers to learn mathematical concepts thoroughly and to be able to adjust to additional changes in application and methods in the future seems paramount.

PURPOSE OF THIS BOOK

This book has been written to supply data which will guide professional educators and mathematicians in the preparation of new mathematics curricula. The application of valid evaluation procedures to both old and new mathematical programs has been emphasized as the most important problem facing curriculum workers. The main purpose of this book is the presentation of evaluation instruments and the suggested use of evaluation techniques in teaching and curriculum improvement work.

This chapter includes reasons for the new mathematics, historical backgrounds, and a statement of problem areas.

REASONS FOR NEW MATHEMATICS

There are four major reasons for changes which are taking place in the new mathematics curricula.

1. The first reason for change has been the new discoveries in mathematics and science. Research mathematicians have developed more new mathematics in the past thirty years than has been developed in the rest of the history of this discipline. Some of the new fields are: linear programming, theory of games, applications of set theory, application of probability theory to such fields as traffic flow and communications, and mathematics for space flight. In addition to the utility in mathematics, theoretical mathematicians are making tremendous advancements in pure mathematics.
2. A second reason for the new mathematics has been the demands from all aspects of American life for professional mathematicians, mathematically trained scientists, and mathematically skilled personnel in hundreds of occupations.
3. The new industrial revolution, caused largely by discoveries in mathematics and science, is the third reason for more and new mathematics. Automation alone demands many thousands of mathematically trained men and women to use digital computers. As machines take over the work done by unskilled labor, the demand for mathematically educated personnel increases.
4. Finally, vast sums of money are being spent on the writing of mathematical materials for classroom use and in the developing of new teaching procedures. These projects are being financed with funds from the government, private foundations, and the publishing industry. Some of the organizations which have given financial assistance are: The National Science Foundation; Carnegie Corporation; Department of Health, Education, and Welfare; and research branches of

military and industrial organizations. Projects funded by these groups have been established throughout the nation, including the School Mathematics Study Group, the University of Illinois Committee on School Mathematics, the University of Illinois Arithmetic Project, the Minnesota School Mathematics Center, the University of Maryland Mathematics Program, Stanford Projects, and many others.

These four causes of the new mathematics have made this discipline the fastest growing and most drastically changing of all the sciences. This rapid growth has created many interesting yet perplexing problems which will demand solutions if the new mathematics is to continue growing and contributing to the mathematical needs of our nation.

HISTORICAL BACKGROUNDS

In order to understand the confusion which surrounds the acceptance or rejection of the new mathematics,[1] one must be familiar with the historical backgrounds of arithmetic in the elementary schools of the United States. Four major periods in the development of arithmetic teaching, including the new mathematics, will be identified. Vestiges of the first three periods are still found in current practice today and constitute areas of friction and disagreement which hinder the growth of new mathematical programs.

The drill theory

Before 1900 nearly all teaching in the elementary school was done by the use of drill. Arithmetic was hard for many children because textbooks were cluttered with difficult and often meaningless problems. The drill theory of teaching was consistent with the prevailing purpose of education. The doctrine of Formal Discipline, that learning consisted of training special faculties of the mind, prevailed. The main purpose of education according to a formal disciplinist was to train the faculties such as memory, reasoning, and will. Faculties could be strengthened by exercise. The most difficult subjects (Latin, Greek, mathematics) were also the most desirable subjects in the curriculum because they provided the most exercise.

---

[1] Throughout this book the word *mathematics* has been used to cover measurement, properties, and relations of quantities, including arithmetic, geometry, algebra, and other aspects included in the new programs. *Arithmetic* is used to describe the art of computation with numerals—the elementary aspects of mathematics. Elementary school mathematics will become the preferred term to describe modern programs.

An attack against the doctrine of Formal Discipline had been launched by William James in his *Principles of Psychology* in 1890. By the turn of the twentieth century, Edward L. Thorndike and Calvin Woodward began making direct measures of transfer effect that proved fatal to formal discipline theories. They found that no learning activity has any great mental disciplining power. There is no such thing as a general transfer of training.

Experimental interest in drill in arithmetic began shortly after the turn of the twentieth century and probably reached its peak by 1920. Edward L. Thorndike was the leading worker in this field and advanced the theory that for every situation or stimulus there is a particular response—the two being associated or tied together by a specific bond (S—R). These theories, pertaining to drill and learning in general, were applied to the teaching of arithmetic. Arithmetic abilities were thought of as numerous specific skills which could be learned best by the application of drill exercises.

Instructional programs in arithmetic were organized so that children would learn all number facts as the basis for efficient work in arithmetic. Number work was systematically presented at each grade level by the use of a textbook. Artificial, rigid standards were set for each class. All children were expected to work up to the standard set for the grade. This, of course, made arithmetic very difficult for the lowest one-third of the class. The average or middle group found the work close to their ability level. However, the bright children (the top one-third) were seldom challenged by this type of program.

During the late 1920's the Field Theories of learning, stemming from the Gestalt psychology, challenged Thorndike's laws of exercise and effect and brought to a close the heavy emphasis upon drill. The Gestalt school of psychology stressed the total organization of learning and the systematic arrangement of the whole rather than the individual elements. The whole must be conceived as more than the mere sum of the parts. Understanding the number system, stress upon problem-solving, and drill after understanding of specific facts were part of a new learning theory.

Unfortunately there are numerous traditional teachers who still cling to the outmoded drill theory of teaching. For them the textbook is still considered the main instructional aid, and children are expected to master the content within the textbook regardless of individual differences. The drill method of teaching also receives support from parents who were taught by this method and who are frustrated by the new mathematics which they can not understand.

Social aspects of arithmetic

Several forces are responsible for the attempts to prune useless and obsolete materal from the arithmetic curriculum during the period from 1890 to 1930 and for attempts to stress the social aspects of arithmetic. During the years when formal discipline was the accepted educational philosophy, textbooks became cluttered with much difficult and abstract content.

The Child Study Movement, which was started by G. Stanley Hall shortly after 1880 and reached its height as an educational movement by 1920, provided much information about the needs of children—including arithmetic.

National committees—the Committee of Ten on Secondary School Studies in 1893 and the Committee of Fifteen on Elementary Education in 1895—recommended that obsolete and impractical aspects of the arithmetic curriculum be eliminated from the elementary school curriculum.

Shortly after the turn of the twentieth century, educators strove to find some method to determine the content for the arithmetic curriculum. Studies were made of the arithmetic used by adults in their vocational and nonvocational daily activities. This approach became known as the *social utility* method. This theory held that arithmetic content should have a clear relationship to life needs of adults. Evidence gathered by questionnaire studies of arithmetic used in many vocations revealed that cube root, abstract mensuration, square root, complex fractions with denominators larger than eighths or sixteenths, and much of the commercial arithmetic could be deleted from the elementary school curriculum. In addition to reducing the content of the arithmetic curriculum, the social utility movement also established beyond any doubt the fact that arithmetic had a social aim. Arithmetic was assured of a place in the elementary school curriculum because of its contribution to effective, intelligent daily living.

The Child Study Movement branched off into several other educational movements after 1925. A large part of the movement was absorbed by the Progressive Education Movement which was founded in 1925. Through the twenty-five or thirty private schools which constituted the beginning of the Progressive Education Movement, two other phases of the social aspects of arithmetic had their beginnings: (1) the incidental program and (2) arithmetic taught as an integrated part of other subjects. These two phases have a direct bearing upon philosophical principles held by some kindergarten and primary-grade teachers today.

The incidental program was based upon the assumption that children have social needs for arithmetic in their daily living. Children will continue to find arithmetic a part of their normal work after entering school, according to this theory. Having had some use for number in pre-school activities, provision was made for informal mathematical experiences whenever there was a need for quantitative thinking. Teachers who favored this program for kindergarten or first grade believed that many children were too immature for systematic instruction. They also believed that instruction was wasteful when given before children were ready for it. Some ill effects from forcing children to learn before they were ready or from imposing burdensome work upon children would result in emotional blocking and dislike for mathematics or even schooling in general.

This type of incidental teaching was commonly found in the kindergarten. Some private schools and experimental schools used incidental teaching of arithmetic in primary grades (K-3). The theories were not implemented in public schools. Closely associated to the incidental approach to the teaching of arithmetic in primary grades was the informal but planned program. Children were taught to use arithmetic as they worked on social studies units dealing with the grocery store, post office, or dairy. In this program there was conscious exploitation of all kinds of situations which occur daily in an elementary school well organized for teaching arithmetic.

Three forces were at work to challenge the social aspects of arithmetic: (1) advancement in the field of child growth and development pointed out the necessity for developmental, continuous learning of new skills (53); (2) leading educators like Paul Hanna (41) were discovering the gaps in the learning of arithmetic by the incidental and informal approach; and (3) educational psychologists led by William A. Brownell (6) were proving that meaningful learning of arithmetic was based upon understanding the structure of the number system. The incidental learning theories, while still useful, were displaced by the meaningful approach to the teaching of arithmetic.

The meaningful arithmetic program

Interest in the mathematical aspects of arithmetic, rather than the incidental or social phases, initiated the meaningful approach to the teaching of arithmetic. This program, which began about 1930, is the popular movement today. Advocates of this approach stressed learning with understanding as contrasted to learning by drill or by examples without understanding. Arithmetic was viewed as a system of understandable ideas, principles, and processes. The basic idea in this approach was to help children develop the ability to do quantitative thinking and to understand the mathematical and practical aspects of arithmetic. The meaningful approach to teaching arithmetic received its original impetus from psychologists who were proving advantages of learning with understanding instead of learning by isolated drill. William A. Brownell, one of the outstanding advocates of meaningful teaching, has identified four aspects of meaningful arithmetic: (1) basic number concepts, (2) fundamental operations, (3) important principles, relationships, and generalizations, and (4) the understanding of our decimal number system (9).

Brownell pointed out that, from the standpoint of the teacher, meaningful arithmetic is interesting to teach. The need to develop understandings is a stimulating experience compared to teaching arithmetic by listening to memorized facts and by administering mechanical drill.

The advantages of meaningful arithmetic to the pupil, according to Brownell (9), are:

1. Gives assurance of retention.
2. Equips the pupil with the means to rehabilitate quickly skills that are temporarily weak.
3. Increases the likelihood that arithmetic ideas and skills will be used.
4. Contributes to ease of learning by providing a sound foundation and transferable understandings.

5. Reduces the amount of repetitive practice necessary to complete learning.
6. Safeguards the pupil from answers that are mathematically absurd.
7. Encourages learning by problem-solving in place of unintelligent memorization and practice.
8. Provides the pupil with a versatility of attack which enables him to substitute equally effective procedures for procedures normally used but not available at the time.
9. Makes the pupil relatively independent so that he faces new quantitative situations with confidence.
10. Presents the subject in a way which makes it worthy of respect.

Meaningful teaching of arithmetic is dependent upon a well-prepared teacher who understands the structure of our number system and who is well versed in modern principles of learning. The classroom organization must provide for a stimulating environment where children can discover the meanings and applications of arithmetic according to their ability. Grouping within the class and careful planning of course content must be provided. Drill, which was so important in the traditional program, still has a place (along with suitable practice) but follows understanding of each new step. Incidental teaching, too, has a place when the teacher recognizes that mathematics has a distinct contribution to make to the solution of problems in other school subjects such as science, social studies, or English.

The new mathematics program

The causes of the new mathematics have been discussed earlier in this chapter. In this section the historical backgrounds of elementary school arithmetic will be extended to include the changes in the elementary school program *recommended by the leaders of the new mathematics.*

The experimental programs such as the School Mathematics Study Group (S.M.S.G.), the University of Illinois Arithmetic Project, and the University of Maryland Mathematics Project have provided the laboratories for preparing instructional materials and the education of a corps of teachers for the new mathematics program. Each experimental center has provided some type of institute or summer session for the writing of instructional materials. These new mathematical materials have been used in public schools which have agreed to work with particular projects. The S.M.S.G. materials, for example, have undergone several rewritings and have been used in a large number of schools.

Publicity for the new mathematics has been provided by the National Council of Teachers of Mathematics and their affiliated state organizations. For the past five years leaders of experimental programs have reported regularly to state and national meetings of the Council and in many instances have provided teaching demonstrations using new methods and materials. Local newspapers and educational journals have given an unusually large amount of space to developments within this area. *The Christian Science Monitor*, as one example, carried a series of articles for ten weeks on "The 'New' Math: What Is It? Who Should Study It?" (21). The mathematicians have indeed had a favorable climate in which to develop and gain acceptance of improvements in mathematics.

Contrary to the statements of leading mathematicians working with the new mathematics, there has been considerable confusion over what constituted the "new program." Each experimental center has emphasized some particular aspect, for example: the Greater Cleveland Mathematics Program emphasized helping children discover mathematical relationships and covered new aspects along with the traditional mathematical material; the Madison Project of Webster College and Syracuse University worked on supplementary material for junior high school grades and then extended their work to include elementary grades; San Diego Metropolitan Educational Study Council concentrated on extending the arithmetic program for advanced pupils in grades 3-6; the School Mathematics Study Group covered most of the new strands with some emphasis upon a developmental program; and the University of Illinois Arithmetic Project worked to assist teachers to present basic mathematical ideas to pupils in an exciting and interesting fashion—including extensive use of the number line.

Gradually during the past two or three years there has been some agreement upon unifying themes or strands involved in the new mathematics. There has been considerable agreement that mathematics programs at all levels of public education should stress the meaningful approach to learning and should be focused on basic concepts underlying mathematics.

Textbook publishers have been more confused than the mathematicians. While some companies have introduced considerable new mathematics into their traditional textbooks, most publishers have had to wait until some agreement could be reached by mathematicians and educators on what content should be included in textbooks for children. The state of California has just recently provided guidelines for the adoption of new arithmetic textbooks for 1965.

The California Curriculum Commissions Advisory Committee on Mathematics had identified the cultural changes in recent years which necessitate a new approach to the teaching of mathematics. These changes involve a new emphasis upon content, introduction of new topics, and a new method of instruction. Pupils are encouraged, according to this committee, to make conjectures and guesses, to experiment, to formulate hypotheses, and to understand (18).

Topics in the mathematics curriculum for kindergarten through grade eight have been designated as: numbers and operations, geometry, measurement, and applications. Also the Committee recommended a planned and conscious development of strands which contribute to clarity, precision, and economy. These strands are: sets, functions, logic, and mathematical sentences. These topics and strands are the same as those developed at the national level by the National Council of Mathematics Teachers. Variations will occur, however, in the interpretation and breakdown of each large topic or strand for instructional purposes.

Changes recommended for elementary schools. The desire to teach more mathematics in the elementary school has prompted mathematicians to recommend introducing simple algebra and geometry at appropriate times throughout the elementary school program. For example, when a child learns $4 + 5 = 9$, he can learn one fundamental postulate—the cummutative law, $a + b = b + a$. In adding $4 + 5 = 9$ we can also add $5 + 4 = 9$. The child discovers these laws as he learns his primary facts. In column addition such as:

$$\begin{array}{r}3\\2\\\underline{4}\end{array}$$

he learns that one can add either up or down the column and get the same result. Thus, $(3+2)+4 = (4+2)+3$. This process involves the commutative and associative laws. Simple principles of geometry are introduced as children work with geometrical figures. Arithmetic taught from an algebraic approach deepens understanding of the properties of numbers and prepares children for the algebra which will come later in the school program. Less time will probably be spent on applications of mathematics. Greater stress will be placed upon underlying principles. The purpose is to educate the pupil so that he understands the basic principles and can make whatever applications are necessary for his time.

Some old topics will be restored to the program such as factoring, least common multiple, and greatest common factor. These topics are involved in fundamental properties of the number system and in connection with literal expressions in algebra.

New topics will include: numeral systems to different bases, such as base two (binary); inequalities; simple language of set theory; and new terminology to sharpen mathematical thinking such as ray, half-line, multiplicative, numeral, and set. Some words will be eliminated. Multiplier and multiplicand will be replaced by one word, *factor*, because the product of two numbers is the same regardless of the order in which they are multiplied. Mathematical sentences such as $3 + 4 = ?$, $? + 5 = 12$, or $\triangle + \square = 10$ will be used. The symbols ($\triangle$ and $\square$ used above) indicate different unknown numbers.

In methodology, emphasis is placed upon discovery and intuitive thinking. The primary aim of all teaching of the new mathematics is to help children understand the fundamental structure of mathematics and methods of mathematical reasoning.

During the next decade there will be continuous experimentation and development of new mathematical programs. No one program will replace the basic arithmetical processes now taught in public schools. There will be, however, many attempts to add the many new strands just discussed in this section to existing programs.

PROBLEM AREAS

The reasons for the revolution in mathematics and an historical background for understanding the confusion surrounding the introduction of new mathematics programs have been briefly described. The purpose for so doing has been to point out problem areas which must be dealt with in curriculum planning, teaching procedures, preparation and use of instructional materials, and the education of teachers for the new mathematics program. These problems are:

1. *The new mathematics and the purposes of elementary education.* The need for new mathematics is based upon new content, demand for mathematically educated personnel in all

aspects of American life, and the necessity to educate more mathematicians and scientists for the space age. How does this need fit into the purposes of the elementary school? Are all elementary school children capable of learning the new mathematics? If elementary children can learn the new mathematics, how much should they be asked to learn in terms of efficient instruction and in terms of the needs and capacities of children? What is the most appropriate mathematical content for elementary school children?

2. *Teacher preparation and the new mathematics.* There are many traditionally oriented teachers in public schools, colleges, and universities who cling to outmoded methods of teaching (particularly the drill theory of teaching). Many do not have an adequate understanding of the structure of our number system. How should these teachers be re-educated so that they can teach the new mathematics? Since public education is dependent upon the financial and moral support from every community, how can parents and lay people be re-educated to support the new mathematics?

3. *Relationship of new mathematics to other school subjects.* Historically, a few elementary schools taught arithmetic by an incidental approach or through planned classroom activities connected with other subjects. These approaches were confined to the kindergarten and to some private experimental schools which had a delayed arithmetic program for primary grades. How much mathematics should be taught as integrated parts of other school subjects such as social studies or science? If less time is spent on applications of mathematics during the regular instructional time, should planned social aspects of mathematics be provided for in connection with instruction in other subjects?

4. *Learning theories and the new mathematics.* The meaningful approach to the teaching of arithmetic grew out of an interest in the mathematical aspects of arithmetic. The mean-

ingful approach to teaching received its original impetus from psychologists who were proving advantages of learning with understanding instead of learning by isolated drill. How can leading educational psychologists be utilized more advantageously in the new mathematical revolution? How can top mathematicians be assisted in their understanding of basic theories of learning which have been used in the teaching of arithmetic for many years? How can modern principles of learning theory be presented to elementary school teachers so that the quality of instruction in mathematics will be improved?

5. *Dearth of evaluative instruments for new mathematics.* The main aspects of meaningful arithmetic (concepts, operations, principles, our decimal system) have been identified. Considerable research work and experimentation with teaching procedures has been carried out in attempts to develop appropriate teaching materials and methods to advance the meaningful approach. What are some techniques which can be used to evaluate pupil understanding of arithmetic? Why has there been such a dearth of tests to measure pupil and teacher understanding of arithmetical concepts?

ORGANIZATION OF BOOK

A critical review of research reports, supplemented by new data collected by the writer, will be given in Chapters II and III. The development of evaluation instruments to measure pupil understanding of basic arithmetical concepts are included in Chapter IV. The writer's point of view on the basic problems identified in Chapter I is given in Chapter V. Chapter VI is devoted to a bibliography of significant research and writing in the field of elementary school mathematics. Tests to evaluate pupil understanding of basic arithmetical concepts for grades three through six have been included in the appendix.

NOTES AND PROBLEMS

1. In 1960 the National Council of Teachers of Mathematics, with financial support from the National Science Foundation, held eight Regional Orientation Conferences in Mathematics in various parts of the United States. These conferences provided information for administrators and supervisors on new mathematics programs. The results of these conferences are reported in the following pamphlet: National Council of Teachers of Mathematics. *The Revolution in School Mathematics.* Washington, D.C.: The Council, 1961, pp. 1-90.
2. For a quick overview of historical backgrounds of arithmetic, read: W. S. Monroe, ed. "Arithmetic," *Encyclopedia of Educational Research.* New York: The Macmillan Company, 1950.
3. A classic for all who are interested in primary-grade arithmetic is William C. Brownell's *Arithmetic in Grades I and II.* Durham, North Carolina: Duke University Press, 1941.
4. One of the recent textbooks (Eileen M. Churchill. *Counting and Measuring.* Toronto: University of Toronto Press, 1961) presents mathematics as a language. She shows how to translate a concrete numerical problem into the symbolic language of calculation. Do you see any connection between this approach and the teaching of introductory work in algebra?
5. If new mathematical programs are being used in elementary schools accessible to you, try to secure information on the type of evaluation instruments used.

II

Pupil Understanding of Arithmetic (Review of Research)

INTRODUCTION

Keeping abreast of modern developments, during the last quarter of a century, has been difficult for many educators and mathematicians associated with or responsible for the teaching of elementary school mathematics. The major contributions made in the teaching of arithmetic during the past twenty-five years have been the defining of the concept of meaningful learning and the carrying out of adequate research to support this development. These developments have been evolutionary in nature and have resulted in gradual modifications.

Since about 1958, moreover, the re-examination of teaching methods, the rewriting of textbooks, the introduction of new mathematics, and the search for new methods of teaching have had truly revolutionary results because of the scope of the changes advocated and the rapidity with which experimental programs have been introduced. The success of the revolution now in progress will be dependent upon wise utilization of what has already been accomplished and upon the extent to which new innovations can be implemented in the elementary school program.

A brief discussion of philosophical and psychological aspects of learning will now be given as a foundation for the review of pertinent research dealing with meaningful instruction in arithmetic.

PHILOSOPHICAL AND PSYCHOLOGICAL CONSIDERATIONS

Social aspects

The revolution taking place in the field of mathematics identifies and accentuates the social aspects of education. In the first place, pupils enrolled in our schools need to learn more mathematics than in any other period of history. This learning can not be gained except by concerted effort of pupils and the schools. Secondly, mathematical education involves a blending of knowledge and culture of the past with the new subject matter of modern mathematics. This, too, is a function of the school. The way educational experiences are presented to children, in order to avoid confusion and to provide efficiency of instructional practices, represents a third aspect of the relation of the school to society. But there remains a fourth aspect, perhaps the most important of all. The school must make value judgments about the kind and amount of education to be provided for children. Certainly children need a balanced mathematical education which will enable them to live and learn in our culture. They must also be assured that the essential spirit of democracy is achieved through the methods and content provided for them.

The pupil

From the vast amount of research in the area of child development has come information necessary for directing pupil learning in elementary school mathematics (81). Development has to do with the sequence of related changes in a pupil from his origin to maturity. As this process takes place, the pupil advances to a higher degree of differentiation and complexity. Certain principles of child development have been listed by educators, psychologists, pediatricians, and others who work in many fields of study. These principles are important to all who are interested in mathematical education:

1. Readiness for learning is determined by the growth of the child as a whole. The child's response to instruction is strongly influenced by his level of development.
2. The child acts and learns as a whole. Intellectual, physical, emotional, and social development must be considered.
3. Mental, physical, social, and emotional traits develop at different rates according to a unique pattern for each child.
4. Development is continuous and is influenced by the interaction of heredity and environment.
5. There are differences within and among individuals. Not only do the rates of development of different children vary but each pupil's rate may vary from one time to another. Differences within an individual are more than half as great as differences among children.
6. There are many acceptable (normal range) patterns of behavior at each age level.
7. Emotions influence mental and physical reactions and activities.
8. Although there are many sex differences, children are also alike in many ways.

Although all of these principles are important, the recognition of individual differences and the provision for these differences is probably the most challenging problem facing the modern mathematics program.

In addition to these principles, which are physical and psychological in nature, there are other basic principles which are mainly sociological. These are major tasks faced by all individuals in our particular society. They have been called "developmental tasks." For example: (1) children are expected to develop certain physical skills needed for ordinary games; (2) wholesome attitudes toward self have to be learned; (3) one learns to get along with age-mates; (4) masculine and feminine social roles must be learned; (5) fundamental skills in mathematics, reading, writing have to be learned; (6) scales of values for morality must be acquired; and (7) attitudes toward social groups and institutions have to be learned. Right now pressures are being placed upon youth to acquire sound education in mathematics and science. Thus, variations in the cultural pattern must be expected and planned for in the educational program of the schools.

Objectives

Many good statements of educational objectives have been prepared by commissions and school systems. One such statement which has been widely distributed is that prepared by the Educational Policies Commission. The purposes of education prepared by the Commission recognize the functions of education as producing informed citizens and changing the behavior of children in relation to persistent individual and social problems. These purposes are made explicit under four large headings: (1) objectives of self-realization, (2) objectives of human relationships, (3) objectives of economic efficiency, and (4) objectives of civic responsibility. Objectives for the mathematics program are included under the first heading—self-realization. This statement, however, merely provides the framework for a detailed list of objectives which must be prepared by educators and mathematicians for each particular school or community.

In the *Twenty-Sixth Yearbook* of the National Council of

Teachers of Mathematics, a statement of objectives for mathematics has been given. It points out the importance of objectives to the total educational program and to the evaluation process.

> . . . Many sets of objectives may be developed; there is no commonly accepted list. We hope that the reporting of our list . . . will not suggest that these objectives are official or complete or the best or the final ones . . . in order to have a framework or reference, a sample list of objectives is presented. . . .
>
> The student should:
>
> have a knowledge and understanding of mathematical processes, facts, concepts;
>
> have skill in computing with understanding, accuracy, and efficiency;
>
> have the ability to use a general problem-solving technique;
>
> understand the logical structure of mathematics and the nature of proof;
>
> use mathematical concepts and processes to discover new generalizations and applications;
>
> recognize and appreciate the role of mathematics in society;
>
> develop study habits for independent progress in mathematics;
>
> develop reading skill and vocabulary essential for progress in mathematics;
>
> demonstrate such mental traits as creativity, imagination, curiosity, and visualization;
>
> develop attitudes that lead to appreciation, confidence, respect, initiative, and independence (77).

Obviously, other objectives dealing with specific aspects of instruction in mathematics could be added to this list. The use of objectives in the evaluative process will be discussed in the next chapter.

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Theories of learning

Principles of learning have grown out of the study of three main theories: association, conditioning, and field psychologies. Each of these psychologies of learning has an explanation of the way the learner proceeds from a need to the achievement of a goal. While a number of points of conflict exist between these theories, there are also areas of agreement and contributions made by each theory. By studying these theories of learning and by reading interpretations of these theories for the learning of mathematics (80), teachers can derive a set of principles to direct their teaching.

Learning, based upon modern theories, would stress problem-solving and effective thinking. The role of the teacher would be to help each pupil examine new learning experiences and to discover the problem or problems to be solved. The learner would collect data, make hypotheses, develop a theory, and finally attempt to verify the theory. Modern theories of learning include: provision for individual differences, greater freedom of activity, learning through experience, meaningful acquisition of skills, and making the most of educational opportunities *now as well as preparing for future living.*

Learning principles appropriate for modern mathematical education follow.

Thus, learning:

1. Involves progressive changes in an individual's behavior directed toward acceptable goals and influenced by experiences.
2. Takes place best when there is a problem and the individual seeks a satisfying solution.
3. Is a function of motive and takes place when there is a need, drive, or goal, the attainment of which will satisfy the motive.
4. Is developmental with time one of its dimensions.
5. Involves a change in organization of behavior.
6. Is a process—not a product.

7. Is a function of readiness to learn.
8. Involves transfer as an important aspect to be sought and directed.
9. Has the development of meanings and understandings as the central problem involved in the process.
10. Recognizes the relationships between the demands of the activity and the characteristics of the child in efficient learning of motor activities.

The foundation areas just discussed provide the guide lines for new developments in elementary school mathematics. Since the major contributions made in the teaching of arithmetic during the past quarter of a century have been the defining of the concept of meaningful learning, sponsors of modern mathematical programs must make certain that these gains are not lost as new mathematical curricula are developed.

RESEARCH ON LEARNING THEORY

In this section, attention will be given to factors which are essential parts of the learning process—readiness, motivation, thinking, concept formation, and transfer.

Readiness

The basic idea involved in readiness is that an optimum time exists for beginning new learnings. To attempt instruction before the child is ready is laborious, wasteful, and often unsuccessful. Gradually, theories of readiness have been clarified and enlarged because of the continuing research work in child development and educational psychology. Readiness principles are considered an important aspect of learning involved in all elementary school subjects. Physical, emotional, social, and mental factors are included in readiness. Both general and specific maturation are important. Other factors influencing readiness for arithmetic and mathematics are: information, previous experiences with these subjects, attitudes, and abilities.

Readiness is important to this book in several ways. Are children who enter kindergarten or first grade ready for arithmetic? Do young children understand certain basic arithmetical concepts so that they are ready for continued work and extension of these concepts? Do children need to understand the meaning and significance of arithmetic?

Primary Grade Readiness. The first major research study dealing with readiness for pupils to begin systematic instruction in primary-grade arithmetic was conducted by William A. Brownell. The study and the findings are appropriate and modern today although the study was reported in 1941! Historically, the reader must consider that during the years of this study (1938-41) arithmetic was not taught systematically in grades one and two. In fact, several programs of instruction were in use: (1) the incidental approach, (2) emphasis upon social applications, (3) traditional practice based upon drill, and (4) concern for meaningful teaching. Brownell showed (with respect to primary number) that no single program of instruction was predominantly favored (5). In his study three important questions were studied: (1) Is the primary-grade pupil intellectually capable of profiting from systematic instruction in arithmetic? (2) If the primary-grade pupil can learn much about arithmetic, should he be asked to do so; is instruction in number wasteful if given in primary grades? (3) If primary-grade children *can* and *should* learn arithmetic from the start of their school career, *what should be the content and form of this teaching?*

In answering the first question (is the primary-grade child intellectually capable of instruction) Brownell showed that children already possessed large amounts of information about arithmetic upon school entrance. Three main levels of achievement were identified: (1) children who knew the skills and concepts of counting and enumerating to 20, could identify through 10 at least, made crude comparisons, knew number combinations to sums of 10, identified simple fractions, told time to the hour, and were familiar with some United States coins; (2) children who did not fully understand skills and concepts but had made reasonable progress in rote counting by 1's to 100, by 10's to 100, 2's to 20 or 30, could make crude comparisons, knew number combinations to 9 or 10, and could read some numerals; and (3) less than a third of the children entering first grade could count by 3's to 30, made crude comparison of objects (same or equal), knew about proper fractions and relative size of fractions, told time to half and quarter hour, knew relative value of United States coins, and knew relative size and units of liquid and linear measures. Brownell then answered the first question with "yes."

The evidence is of two kinds. First, inferential evidence shows that children entering grade one already possess much arithmetic, and if properly taught, they can learn much more in primary grades. Second, Brownell's own research showed unmistakably that if primary-grade children were taught arithmetic, they would learn it. He cautioned educators to interpret learning and mastery as the end-product of long periods of experience and practice. Primary-grade children *made progress toward learning and mastery* of many arithmetical concepts.

The second question (should the child be asked to learn arithmetic if he is capable) was answered by showing individual and social need for arithmetic. Brownell referred to the studies of Woody (5), showing that 83 per cent of the parents in one sampling of 164 homes stated that home instruction was given in rote counting. Other aspects of home instruction included: 54 per cent taught counting to 100; 78 per cent taught children to count objects; 87 per cent taught coin recognition; 68 per cent taught the value of money; two-thirds taught reading and writing numbers and simple verbal problems. The conclusion drawn from this study by Brownell is that someone must assume the duty of teaching arithmetic to children. "Mere attainment of more birthdays cannot be expected to bring the needed increase in number knowledge, apart from directed experience" (5). His position is then clearly stated:

> Research has shown that school entrants already know much about number; the inference is that they can learn more; society requires that children must know arithmetic; nothing is gained, and much may be lost, if the school delays to later grades the discharging of its obligation (5).

Answering the third question (what content to teach in primary grades) was extremely difficult because of the absence of good programs and reliable research. Brownell hypothesized that if arithmetic was adjusted to pupil interests and capacities, first-grade children could and would extend their number knowledge happily, intelligently, and usefully. He went ahead to test this hypothesis by showing what happened when nearly 500 children in grades one and two were provided with systematic instruction in arithmetic. The children in his research study did make substantial progress of a measurable kind. Brownell was the first to point out shortcomings of this program and to note that there were different kinds of arithmetic and different kinds of systematic instruction. Arithmetic programs during this period (1940) were evaluated according to standards of speed, accuracy in computation, and conventional problem-solving. His program measured these factors and added *meaning* and *significance*. He defined these words by saying: "An idea or skill is meaningful to the degree that it is understood. An idea or skill is significant to the degree that the values are known and to the degree that it is used" (5). But the answer given on what to teach was classic and remains as true today as when it was written in 1940.

> At whatever point arithmetic instruction is begun, *whether in grade one, or grade two, or later, children should see both sense (meaning) and value and usefulness (significance)* in the number they encounter. If arithmetic experiences cannot be found to satisfy both of these requirements in Grade I, then arithmetic should be deferred (5).

Later in this chapter the writer will deal with research on thinking and concept development. The research work of Churchill, Piaget, and others has been summarized. These studies all have a direct bearing upon the discussion of readiness. Some of the main findings, therefore, should be reported at this point:

1. In early work with number, certain ways of thinking which are connected with the nature of number and mathematical symbolism must be developed.
2. Churchill stresses the importance of the "operational character of knowledge." In early childhood the bodily operation is enormously important because very little information provided in verbal form can be translated into knowledge at this age.
3. Readiness for learning to deal with numbers with understanding depends upon the maturity of certain mental abilities, favorable physical and mental experiences, and the ability of the teacher to recognize and use moments of readiness.

The Committee of Seven (1928) made a detailed study of starting systematic arithmetic in different grades. About one-third started arithmetic in grade one, another third in grade two, and the rest in grade three. In all but one (column addition) of the twelve arithmetical processes where tests were given to the three groups, the first group that started arithmetic earliest surpassed the other groups. The group starting arithmetic in second grade surpassed the group starting in the third grade. Washburne's Committee concluded that the uniform superiority of the pupils who began arithmetic in the first grade over those who began in the second grade and third grade occurred because they had an earlier start (87).

General Readiness Factors. Other factors influencing readiness are information, attitudes, and abilities learned through experiences. Teachers should not wait for readiness to be achieved. Experiences which contribute to the growth of arithmetic readiness should be provided for each new aspect according to the needs of the pupil or the class. Readiness is a factor in successful work in arithmetic at each grade level. In

1949 William Brownell and Harold Moser pointed out the importance of readiness in their research studies dealing with subtraction in grade three:

> . . . If children have the necessary ability, knowledge, and understanding, they are ready. Their readiness is determined, not by their age or by their grade, but by the kind of arithmetic they have had. By manipulating their arithmetic backgrounds we may shift the placement of any arithmetical topic over a range of several years and of several grades . . . (10).

A few years later, 1951, William Brownell did research work which showed the importance of arithmetic readiness as a practical classroom concept. He asked two important questions:

> (1) Just how ready are children in typical classrooms when the teacher determines readiness on purely practical grounds? and (2) To what extent are children equipped with the arithmetical skills and facts essential to economic learning in connection with a new process? (6).

Arithmetic readiness was studied (fifth grade) in relation to a single process or group of skills—introduction to division by two-place divisors. His findings were:

> (1) . . . the present investigation shows that considerable flexibility should be allowed in the grade placement of the topic in question. The practice of assigning the topic to a particular grade for a school system as a whole is of doubtful validity, for the different classes in that grade vary so widely in ability as to make such uniformity impractical. The practice of assigning the topic more precisely to a given time in a particular grade is even less defensible.
>
> (2) . . . Practices commonly employed in the classroom are unsatisfactory as a means of determining arithmetical readiness. For twenty classes in this study many children were not ready for two-place division when called upon to start the topic.
>
> (3) In order to assume real readiness for two-place di-

visors, instruction must be individualized to a greater extent than is customary. One method is grouping within the class (6).

Readiness for learning arithmetic must be a particular kind of readiness. In primary grades, children must be directed toward certain ways of thinking and the use of mathematical language which are connected with the nature of our number system. Experiences with number obtained before coming to school are repeated, enriched, enlarged, and unified into a meaningful structure. The teacher uses a variety of visual aids and concrete materials to build readiness so that children will develop adequate understandings so essential before writing numbers and learning of combinations takes place. In the next chapter specific techniques for evaluating pupil understandings as an aspect of primary-grade arithmetic will be given.

Children in intermediate grades also need a particular kind of readiness. William Brownell's study on readiness for division, just reported, shows the importance of evaluation of pupil's readiness for each new arithmetical process. The teacher must make certain that basic arithmetic skills have been developed earlier. Other aspects of readiness should be considered: stimulating pupils to work on problems leading up to a new step in order to introduce new learning; building background for new concepts such as decimal notation; providing a desirable mental set for new material or for daily work; and enabling children to discover the new step or method which needs to be learned.

Another aspect of readiness, the mental maturity needed to work different arithmetical processes, was studied by Carleton Washburne and his Committee of Seven on Grade-Placement in Arithmetic. The investigation involved 225 schools in sixteen states and continued for over ten years. Reports made by Washburne in 1928 and 1930 are representative of their work (86), (87). Recommendations of the Committee were based upon the retention of new learnings of children who did not have considerable knowledge of a topic but who achieved a fairly adequate mastery of prerequisite topics and, in some cases, of prerequisite concepts. Time and method of teaching were controlled. The Committee of Seven specified the number of minutes per day and the number of weeks that a topic was to be taught. General teaching outlines were prepared, indicating the methods to be used.

The Committee plotted the average scores made by children on the retention tests against their mental ages and then found the percentage of children making a score of 80 per cent on the retention test against their mental ages. Thus, a "minimum mental-age level" at which three-fourths of the children achieved scores of 80 per cent or more correct was found. For example, the Committee found the addition facts (sums 10 and under) could be learned according to this standard when a child had a mental age of 6.5, the subtraction process required a mental age of 8.9, multiplication facts 10.2, and long division process 12.7.

The findings of this Committee have had many adverse effects upon arithmetic teaching. The study was based upon the drill-type teaching and a very unrealistic view of the meaning of mental age. When arithmetic is taught in a developmental way with a meaningful approach, the standards established by the Committee become ridiculous and unsatisfactory as a basis for discovering when children can best be taught various arithmetical processes. The findings of David Wechsler substantiate the inadequacy of mental age as a base for readiness or grade placement:

> . . . mental age is considered to represent an absolute level of mental capacity, so that an M.A. of 6 means the same irrespective of whether it is obtained by a child of 6, 10, or 16. . . . Instructors in psychometrics have devoted many hours to teaching their pupils first that the M.A. is a measure of mental level (an M.A. of 7 is a mental level of 7 in all children attain-

ing this rating) and then teaching them that this is not true after all (a 5-year-old with an M.A. of 7 has not the same kind of mind as a 10-year-old with an M.A. of 7) (91).

Thus, readiness broadly interpreted has an important role in the determining of appropriateness of new mathematical learnings as well as influencing pupil progress at each stage of development for this subject.

Motivation

The importance of motivation in every learning activity has been recognized by educational leaders and mathematicians who accept a modern philosophy and psychology of education. The *1960 Yearbook* of the National Council of Teachers of Mathematics repeatedly emphasizes goal-centered learning of mathematics (78). Howard F. Fehr in a recent statement of philosophy of arithmetic instruction (33) states:

> . . . that without motivation and drive, no real learning takes place. So we try to develop the type of environment, the type of stimuli that help the child to motivate his learning. When motivation is present, then self-instruction takes place.

Ample evidence exists to show that in the past arithmetic has been taught in a mechanical, routine, drill-centered way. The results of this type of teaching are found in poor attitudes toward arithmetic (30), failure for many students, and the choosing of other subjects for advanced study in secondary schools, colleges, and universities.

One study of Margaret Buswell is indicative of research work being conducted on the relationship between the social structure of the classroom and the academic success of the pupils. This author concludes her research by stating:

> . . . When we consider a classroom of boys and girls in either the early grades or the upper grades, it may be said that in general those who are succeeding in their school work will

> also be succeeding in their social relationships with their peers . . . (17).

The importance of motivation to all cognitive activity is pointed out by Jean Piaget in his writing on the "Genetic Approach to the Psychology of Thought":

> . . . The formal aspect of thought makes way, therefore, more and more in the course of the development to its dynamic aspect, until such time when only transformation gives an understanding of things. To think means, above all, to understand, and to understand means to arrive at the transformations which furnish the reason for the state of things. All development of thought is resumed in the following manner: a construction of operations which stem from actions and a gradual subordination of formal aspects into dynamic aspects . . . Bruner has said that I have called disequilibrium what others describe as motivation. This is perfectly true, but the advantage of this language is to *clarify that a cognitive or dynamic structure is never independent of motivational factors.* The motivation in return is always solidary to structural (therefore cognitive) determined level. The language of the equilibrium presents that activity which permits us to reunite into one and the same totality those two aspects of behavior which always have a functional solidarity because there exists no structure (cognition) without an energizer (motivation) and vice versa (59).

The psychologists, Anderson and Gates, in a discussion of the general nature of learning, have pointed out the importance of the process of learning—including motivation: "The process established and the particular way of learning with its consequent meaning for a way of attacking new problems may be as important to an individual as a particular product" (2).

Churchill points out the role of the teacher and summarizes the importance of motivational factors as well as the total learning process when she states:

> . . . learning only takes place when the learner is responding actively to the experiences offered to him. This is true of all learning whether by adults or children; whether we are considering the learning of motor skills, the development of acceptable modes of social response, or the acquiring of new knowledge. . . . From this it follows that unless the teacher succeeds in gaining the full involvement of her children in what is going on, the physical provisions she has made will be wasted. For full involvement children must find the environment exciting; one which stimulates their curiosity and desire to experiment and find out things. They must find a challenge in it, and the results of their activities deeply satisfying. But further to this they must feel at ease in the environment, free to enjoy themselves, and other people, free to get involved in learning tasks (22).

Finally, Jerome Bruner, in his book, *The Process of Education,* points out the need for motivation in all of education:

> . . . Motives for learning must be kept from going passive in an age of spectatorship. They must be based as much as possible upon the arousal of interest in what there is to be learned, and they must be kept broad and diverse in expression. The danger signs of meritocracy and a new form of competitiveness are already in evidence. Advanced planning can help (13).

RESEARCH ON THINKING AND CONCEPT DEVELOPMENT

Research on the development of children's thinking includes investigation of perception, memory, ability to concentrate, solving problems, and reasoning ability. Studies dealing with reasoning, problem-solving, and concept development are pertinent in the discussion of understanding mathematics.

According to Russell (64), who has summarized scientific studies on children's thinking, general agreement exists among child psychologists that the infant and the young child first learn to perceive accurately and to judge in terms of the immediate physical environment. The evidence in research on the child's ability to reason about problems is not so definite or conclusive. Reasoning ability does not appear suddenly at two, four, or more years of age. The ability grows gradually and, with some irregular advances in certain areas, probably continuously.

Extensive research on the development of children's thinking has been done by Piaget, working at the Universities of Paris and Geneva. The importance of Piaget's work on the description of developmental aspects of children's thinking has not really penetrated the United States. Wallach (84), commenting on this neglect, feels that the studies of the Geneva group contain a close blending of empirical description with theoretical speculation. This makes it difficult for research workers to disentangle empirical description from speculative theory.

Research to verify, clarify, refute, and extend Piaget's work has been carried out extensively by English scholars. Because of this fact, the writer has drawn upon the work of Churchill, Dienes, Lunzer, Lovell, and other British researchers.

Wallach (84) points out that Piaget has data supporting the early statements of Watson and Skinner to the effect that thinking involves internalized behavior. Churchill (22) reminds us that Piaget's primary interest for the last forty years has been in the genesis of mental operations, and that he is the first psychologist to carry out a systematic study of concept formation from the genetic point of view.

According to Churchill (22), Piaget believes that thought activities may be analyzed in terms of groups or systems of operations. These groupings are relational systems which have certain defining properties. Three of these properties are: (1) *composition*—any two-unit operations can be combined to produce a new unit; (2) *reversibility*—any two units which have been combined can be separated again; (3) *associativity*—the same result may be obtained by combining units in different ways. Each of these systems is connected to actions which can be internalized, reversed, or co-ordinated into patterns of thought. The structures used by a child early in life are different from those constructed and used later. Maturity is shown in the way a child demonstrates the use of the three properties listed.

The stages of mental structuring, corresponding to the three types of development just discussed, are: (1) sensori-motor group structures, (2) concrete operation groupment structures, and (3) formal mental structures. Each of these will be discussed.

"Sensori-motor Structures." During this first stage, the child can only perform actions. At first there are no operations because the child cannot internalize activities. But gradually notions of objects are formed, and he learns to construct the idea of an object lying beyond the field of vision. Piaget used a watch to demonstrate this type of mental structure. A watch, with which he and a child had been playing, was hidden. At first the child lost interest if the watch was covered completely. Later the child learned to search for a hidden object. By the age of two the child had formed certain behavioral concepts of a permanent object. This development was made possible because of the organization of the child's perceptual experience and the increased effective co-ordination of bodily movements. The child co-ordinated sensory data and movement information into *sensori-motor group structures*. This represents, to Piaget, the beginnings of intelligence because: (1) the child acted in a certain way in response to a sensory impression—*properties of composition;* (2) he looked for an object which he had seen hidden, going back to the starting point of the activity—*reversibility;* (3) he changed his movement response to a known object—*associativity.*

"Concrete Operation Groupment Structures." This repre-

sents Piaget's second stage. The stage is much longer than the sensori-motor group and has three phases: (1) "preoperational thought," (2) "intuitive thought," and (3) the "concept of conservation."

Preoperational thought is found from about two to four years of age. The child's actions are internalized because of the use of imagery to imitate an action. The beginning of symbolic behavior is noted, and the child learns to use language. Piaget stresses the egocentric attitude which the child has during this period. The small child finds it difficult to distinguish between inner experiences and external reality. Thinking has started because the child uses symbols to represent objects and experiences. Piaget believes, however, that this thinking is preoperational because the internalized actions are not reversible, and conservation is only understood at the sensori-motor level.

The next phase, according to Piaget, is *intuitive thought*. This develops in children between four and seven years of age. While concrete operation groupment structures are slowly becoming organized, the child's thinking is still tied to perceptual factors. The child's structures are rigid and irreversible. Piaget demonstrated this stage with the use of a set of beads. First the child was asked to put out a set of beads corresponding to a model set. He was asked to say how many beads were in each set and agreed that the sets were the same. Then the arrangement of the model set was changed so that it occupied more space. The child was then asked whether the sets were still the same. A child with irreversible mental structures will say that the set spread out is more than the other set because it is wider—the usual response of a five-year-old child. The child's judgment is based on immediate, intuitive perception rather than upon what has been done with the beads.

The *concept of conservation* is achieved when the child becomes capable of moving in reverse. The child sees that moving a number of beads to make a longer line or to occupy more space than an equal number of beads does not change the number or value of the beads. He understands the numerical value of the set. The child now has the concept of conservation of a sum.

Piaget believes that when a child has the concept of conservation of a sum, the concrete operation groupment structures are becoming organized. The child's actions are internalized, and this enables him to work with concrete operations, classification, and seriation. By age seven, Piaget believes that children can work with concepts of number as equivalent and orderable, time, and space. Inhelder (45), however, has shown that a child must be ten or eleven years of age before he is capable of reversible reasoning in connection with weight and volume.

"Formal Mental Structures." Earlier operations involved reasoning about things and events. Now, in formal mental structures, the pupil is capable of reasoning about these concrete operations themselves. This period is reached, according to Piaget, in early adolescence. The adolescent is able to form hypotheses, make assumptions, and draw conclusions.

Conclusions on children's thinking

What, then, are some implications of Piaget's research for teaching mathematics with understanding?

1. The most striking conclusion is that the child's basic cognitive categories for interpreting physical reality (thinking, reasoning, and understanding) are the product of slow and painstaking construction.
2. Piaget believes that development proceeds through a series of clearly defined stages, each having its own particular type of mental operation. He further believes that (while recent research may show that the general rate of progress through the stages may be accelerated or retarded by experience, culture, or other factors) the sequence of stages will be invariant. Maturation, according to Piaget, takes this form.
3. While we can accept much of Piaget's work, much more attention needs to be given to individual differences in growth and development. Recent research by Bruner, Dienes, Lovell, and others shed

light on differences between children in the way they deal with experiences and develop concepts. These will be reported on in the section on concept development which follows.

4. Certainly Piaget's work suggests that the teacher must provide children with experiences which will enable them to operate constructively and gradually to become aware of the relationship between things defined in the language of mathematics. The teacher must search for the teachable moment to enable children to develop mathematical concepts.

Having spent considerable time on children's thinking, our attention will now be turned to a discussion of mathematical concepts and their development.

Concept formation

John Dewey, in his early work on thinking, has shown that concepts enable one to generalize, to extend and carry over our understanding from one thing to another. "Since concepts represent the whole class or set of things, they economize our intellectual efforts. The concept shows that a meaning has been clearly established and will remain the same in different contexts (25)."

Lovell, an important British writer in this field, identifies the sequence in concept development as "perception, abstraction, generalization." He defines a concept as "a generalization about data which are related; it enables one to respond to, or think about, specific stimuli or percepts in a particular way. Hence a concept is exercised as an act of judgment" (49).

Carefully and gradually children must be taught the language and symbols of mathematics. This language enables children to communicate in a particular, exact way with their peers and the teacher. When properly taught and when thoroughly understood, mathematical concepts may be used to solve new problems, or they may be combined with other concepts and used to solve problems. Occasionally children, particularly while learning to use newly acquired concepts, may not be able to verbalize the concept being studied. In fact, considerable evidence exists to show that premature verbalizing of generalizations may be harmful to some children. This particular point will be discussed later in this chapter under transfer of learning.

In the learning of concepts, Carpenter studied the difference between functional learning and rote learning. He concluded:

> . . . functional learning of concepts is more efficient than rote learning when measured by retention and ability to verbalize meanings of learned concepts. Concepts that refer to classes of material objects are more thoroughly understood when the student has an opportunity to manipulate and study the objects than when only factual information is given by lectures (19).

Betz also points out the crucial need for nonmechanistic, meaningful teaching processes based on motivation, insight, and purposeful application (3).

Dienes conducted research which helps to answer the question about Piaget's stages of development. He found that some people must formulate their concepts very explicitly and exactly before they can use them while others can think in terms of the same concepts with equal efficiency but with only a small amount of such explict formulation. He concluded that it was only through constructive activities with structured material that the child can be led on to analytical thinking. "Constructive exercises have to precede analytical insights. It may be that some children will never reach the analytical stage; the great challenge to the teacher is that of devising ways and means of making the whole of mathematics available to such children (27)."

Churchill concludes, after referring to the research findings of Piaget, Dienes, Wheeler, and others that "the eventual attainment of the number concept depends on the opportunity the child is given to handle real things rather than on the committing to memory of number bonds through frequent exercises with number cards (22)."

Visual Perception and Imagery in Concept Development. The visual and manipulative approach used to aid children in developing mathematical concepts has been open to attack on the grounds that the method does not allow children to see number relationships and may be quite rigid if inadequately used. A variety of number apparatus has been devised by Stern and Cuisenaire which may meet some of the objections leveled at this type of equipment.

Both Cuisenaire and Stern methods provide for children's rearranging of materials, working on concepts which are mathematically sound, and relying only in part upon visual perception and imagery.

Stern (73), through the use of cubes and other manipulative aids, believes that materials are not just for manipulation. These materials supply certain physical structures which lead to mental structures. Later in the learning period the child must be able to visualize the meaning of numerals without the aids being present. Stress is placed upon the child's *measuring rather than counting.*

Cuisenaire materials consist of a number of rods of ten different colors, varying in length from 1 cm. to 10 cm. Each rod is 1 cm. by 1 cm. in cross section. Rods of one color are the same length. Number concepts are learned by seeing and using the rods. As in the Stern materials, no counting takes place because the rods are not marked off into lengths. The child must remember the length and color of each rod. The child is encouraged to discover mathematical relationships by using the colored rods.

Lovell interprets Piaget's views on visual perception as follows:

> . . . Piaget believes that mathematical concepts are not derived from materials themselves, but from an appreciation of the

significance of the operations performed with the materials. The concepts and the ability to maneuver them in the mind, he considers are built up from using concrete material, but are independent of actual materials used (49).

H. Van Engen and E. Glenadine Gibb have emphasized the importance of physical or concrete experiences which are patterned and lead to the development of concepts; these concepts are themselves a pattern of physical elements which are similar to the concrete experiences. They recommend that close attention be given to two realms of structure: (1) a physical structure (the sequence of events and arrangements of objects) and (2) a physical structure (that which causes a given situation to make sense so that everything fits) (83). Both visual aids and verbal problems must be evaluated to discover how closely they fit the mathematical ideas which the pupil is expected to structure.

Certainly the cautions given by Van Engen and Gibb seem sound and practical. Research studies in England lend support to the evaluation of instructional materials to see if they are suited to the child's developmental level and to appraise the mathematical concept being studied. Dienes has found that if a concept involves a number of variables, all variables must be dealt with in order to ensure understanding (27). Lovell found that while Dienes' theory was correct, when working with seven- to eleven-year-old children the teacher should be careful while using apparatus involving a number of variables (49). This study confirmed similar findings of Inhelder and Piaget.

The question about using teaching aids before children are ready for a new mathematical concept is often raised. A partial answer to this problem is implied in the statement of Van Engen and Gibb when they recommend that the teacher pay close attention to both physical and psychical structure. Lunzer, in a study dealing with an investigation of Piaget's work on concept formation, expressed the view that it may be worth while for a teacher to go through processes the child does not quite understand rather than wait until the solution is readily seen. This procedure increases the child's familiarity with the material and the kind of reasoning involved (50). The studies of Brownell (6), discussed earlier under the heading of readiness, point out the necessity to discover when each individual is ready for new arithmetical learnings. This problem will be dealt with again in the next chapter when suggestions are made for evaluating pupil understanding of each basic mathematical concept he is expected to learn.

Summary on concept development

Lovell concludes his book on concept development by showing that mathematical concepts develop slowly. He believes that concepts do not develop in an "all or none" fashion but are at first vague and hazy. Gradually concepts grow in clarity, in width, and in depth with maturation and experience. The rate of development, he feels, is dependent upon the quality of the brain mechanism of the child, upon his motivation, and upon the cultural milieu. The two most important factors likely to affect the development of mathematical concepts are: (1) the mathematical understanding of the teacher, and (2) the climate of opinion in which the child is reared (49).

David Russell, in his excellent book on *Children's Thinking*, summarizes the evidence he has gathered on mathematical concept development as follows:

> . . . mathematical concepts are closely related to the child's perception of number, space, time, weight, and related factors. In their earliest stages, accordingly, they are dependent upon many opportunities to manipulate and explore—an environment rich in blocks of different sizes and colors, toys that go, equipment that can be pulled, pushed, reconstructed, and climbed. Such experiences seem to be a necessary background for any verbalization of mathematical concepts and more exact use of them in appropriate conversational situations.

> Most children seem to go through a rather similar sequence in the development of their mathematical ideas, although wide differences exist at any one age level (64).

Thus, from the abundance of research evidence now available on the development of children's mathematical concepts, certain general principles can be formulated to guide instructional practices:

1. Concept development is the central and most important aspect of directing children's understanding of arithmetic and mathematics. Concepts carefully learned enable the child to generalize and to extend understanding from one aspect of arithmetic to another.
2. Carefully and gradually children must be taught the language and symbols of mathematics. This mathematical language enables them to communicate in a particular, exact way and to develop concepts necessary for successful work in mathematics.
3. Children learn through exploration and discovery. But the role of the teacher in this process is crucial and exacting. For each child and for each stage of development, the teacher must create an appropriate environment which will evoke discovery, impel thinking, and extend mathematical experiences. Growth toward successively higher levels of mathematical thinking and abstraction is the main goal of this guidance.
4. The brilliant work of Piaget helps to identify the developmental levels of mental growth in children and accentuates the importance of ability, maturation, and environment in all learning experiences. Children do, indeed, go through similar sequences in the development of mathematical ideas although wide differences do exist at each age level.
5. Sensory-perceptual materials are necessary for helping children acquire number concepts. However, improper, undirected, and premature use of these aids is harmful to children. Visual aids and verbal descriptions used in concept development must be evaluated to see if sound mathematical ideas are being developed and to see if they are suited to children's developmental levels.
6. Piaget's research has poignant meaning for use of sensory-perceptual aids because he shows that mathematical concepts are not derived from materials themselves, but from an appreciation of the significance of the operations performed with the materials.
7. The development of mathematical understanding and skill at each successive level of pupil progress in school is dependent upon the principles of concept development just presented.

Transfer of learning

The first important evidence on the successes and values of meaningful teaching came from studies which showed the transfer values when arithmetic was taught so that children understood the processes. Evaluation techniques used in these early studies consisted of standardized achievement tests. Pupils taught with drill methods could make high scores on tests which were based upon computational skill. When investigators began to search for retention of newly acquired learnings and transferability of these learnings to new problem situations, evidence began to pour in to show the values of insightful learning. A brief review of research and writings showing the transfer value of meaningful teaching follows.

A major start toward understanding the value of meaningful teaching and learning was made by T. R. McConnell in 1928. His investigation dealt with the learning of the 100 basic addition and the 100 basic subtraction facts. Two methods were used in the controlled experiment: (1) repetition of stimulus-response connections (authoritatively identified) with no attempt to invest them with meaning, and (2) stress upon discovery of truth and meaning—learning through insight, rather than sheer repetition. While evaluation techniques were not adequate, the results showed that the mechanical, drill-oriented group excelled on immediate and automatic response to the number facts as measured by tests with limited administration time and speed instructions. The meaningful-teaching group excelled in tests which put a premium on deliberate and thoughtful responses, and those with longer administration times (51).

Orata summarized 99 studies on transfer in 1928. He showed that the studies revealing little or no transfer dealt with content difficult to generalize. There was a steady increase of transfer with material easy to generalize. By 1941 Orata showed that transfer was revealed by 80 per cent of the studies, but transfer was not an automatic process. Transfer had to be worked for. The amount of transfer is conditioned by factors such as age, mental ability, time intervals between learning and transfer, accuracy of learning, meaningfulness of learning situation, method of study, and provision for reconstruction of experience. Orata reported that the evidence available disproves Thorndike's theory of identical elements (54). When the theory of identical elements of Thorndike became obsolete, the role of insight and generalization became thoroughly established. The work of Hendrix, reported later in this section, is particularly appropriate on this last approach to transfer.

C. Louis Thiele's study made in 1938 is typical of a large number of experiments which followed McConnell's work and which attempted to measure pupils' ability to use newly acquired learnings in novel situations and to generalize. Two methods of teaching were used: (1) the drill method which stressed repetition of the number facts with the use of manipulation of concrete objects and (2) the generalization method which involved work for speed and accuracy as in group one but which depended upon generalizations (seeking likenesses and differences in number situations) rather than memory for recall and use of facts. On the final test (30 items involving transfer of training) the second group (generalization method) averaged 10.00 items correct; the drill-method group averaged 6.07 correct. The superiority of group two was especially pronounced in the learning of difficult addition facts (82).

One of the most significant studies of transfer of training was made by Gertrude Hendrix in 1947 and repeated and confirmed by Haslerud (42) in 1958. A study was made to test the value of the unverbalized awareness method of learning a generalization. Her results were:

> . . . (1) for generation of transfer power, the unverbalized awareness method of learning a generalization is better than a method in which an authoritative statement of the general-

> ization comes first; (2) verbalizing a generalization immediately after discovery does not increase transfer power; (3) verbalizing a generalization immediately after discovery may actually decrease transfer power; (4) the fact that it is the intermediate flash of unverbalized awareness that actually accounts for the transfer power is a new and startling proposition in learning theory (43).

Brownell and Moser provided clear evidence regarding the relative transfer values when the skill of borrowing in subtraction was taught meaningfully and mechanically: "When subtraction examples were understood and rationalized by pupils, there was greater transfer to untaught examples than when initial instruction had been mechanically associative" (10). Subskills in borrowing were also learned and taught more easily and effectively when the initial instructional experiences were rational and meaningful.

The psychologists Anderson and Gates, in a chapter dealing with the general nature of learning, conclude:

> It seems clear that learning which is cognitive in process and control is more likely to have varied utility than learning which is mechanically associative. The most brilliant illustration of this concept is found in the development of the "meaning theory" for teaching arithmetic (2).

Research has shown that transfer is not automatic. Transfer is dependent upon teaching which utilizes meaningful materials, methods, and problems. Independently derived principles, encouraged by freedom for pupils to discover and work out their own solutions to problems, are more transferable than principles which are stated for children. Transfer takes place best when learnings involve common things easy to generalize and when pupils have strong intelligence.

CONCLUSIONS ON EDUCATIONAL WRITINGS AND RESEARCH DEALING WITH UNDERSTANDING ARITHMETIC

The importance of educational philosophy, child development, clearly defined objectives, and sound principles of learning have been stressed in this chapter. The meaningful approach to teaching arithmetic is based upon modern developments and discoveries in these foundation areas.

A particular kind of readiness for learning new arithmetical skills and concepts is needed for each age group. Younger children must be given experiences which will help in the use of mathematical language and thought processes. A variety of visual aids and concrete materials is needed to develop adequate understandings of each new aspect. Children in intermediate grades must know arithmetic skills developed in earlier grades and must have the background for new concepts. The maturity of the children, learning abilities, and previous experiences necessitate careful provision for individual differences.

The importance of motivation to every learning activity has been stressed. There is ample evidence to show that in the past few decades not enough attention has been given to the use of motivation in the teaching of elementary school arithmetic.

Since concept development is the central and most important aspect of directing children's understanding of arithmetic and mathematics, a list of principles was formulated to guide instructional practices.

Teaching arithmetic so that pupils understand each new aspect aids in transfer of learning. The importance of the unverbalized awareness method of learning a generalization to transfer power has been stressed. Learning which is cognitive in process and control is more likely to have varied utility than learning which is mechanically associative.

NOTES AND PROBLEMS

1. While the educational writing and research unreservedly support the concept of meaningful learning, there has been a gap between theory and practice in the teaching of elementary school arithmetic. How would you bring about constructive changes in the teaching of arithmetic?
2. The author has listed ten principles of learning for the teaching of elementary school mathematics. Show how you could apply each of these in teaching at one or more age or grade levels.
3. Arithmetic readiness has been discussed in this chapter from the standpoint of necessary ability, knowledge, and understanding of mathematical concepts. What evidence can you find to show that pupils are ready for the new mathematics? How should readiness for new mathematical learnings be established?
4. What are some of the implications of Piaget's view that mathematical concepts are not derived from materials themselves?
5. How would you apply Gertrude Hendrix's theory (unverbalized awareness method of learning a generalization to transfer power) in the teaching of elementary school mathematics?

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III

Evaluation Procedures (Review of Research)

The purpose of this chapter is twofold: (1) to review research and writing dealing with the evaluation of pupil understanding of basic arithmetical concepts; and (2) to identify weaknesses in the evaluation of pupil growth in understanding arithmetic.

EVALUATION OF UNDERSTANDINGS

Evaluation is the process used to determine the amount and quality of pupil understanding of and achievement in arithmetic based upon clearly defined objectives. This concept has evolved gradually during the past two decades. The evaluative process involves three important elements: (1) the comprehensive range of objectives are evaluated rather than just subject-matter achievement; (2) a variety of techniques is used to secure data on pupil progress; (3) emphasis is placed upon the child's total personality development. This over-all view of evaluation in arithmetic has by no means been widely used by teachers, curriculum workers, or administrators.

One of the first clearly defined statements of the evaluation of learning in arithmetic was made by William Brownell (5) in 1941. He pointed out the need for uniting measurement and evaluation with instructional practice—"instruction and evaluation go hand in hand." The outcomes to be evaluated and the purposes of evaluation must be carefully described. As far as the evaluation of mathematical understandings was concerned, Brownell stated that "exceedingly little has been done either informally or systematically to find practicable and valid procedures for evaluating mathematical understanding." Unfortunately, this statement is generally true today. While considerable progress has been made in many informal studies to measure pupil understandings of arithmetical concepts, no systematic or widespread development has been made in this area. A brief review of studies showing the inadequacy of standardized tests to measure mathematical understanding is appropriate at this time.

Standardized tests and pupil understanding of arithmetic

As early as 1934, Brueckner (11) showed that the unit scales of attainment did not measure all arithmetical outcomes. Using vocabulary and quantitative relationships tests with 453 children in grades 4A to 5B, he obtained coefficients of correlation of .361 between vocabulary and computation, and of .522 between vocabulary and problem-solving. The quantitative relationships test correlated .576 and .661 with computation and problem-solving.

Spainhour (70) confirmed Brueckner's findings. He prepared tests of mathematical understanding with reliability coefficients of .901 for grades four and six. These tests were administered along with the New Stanford Reasoning Arithmetic Test and the New Stanford Computation Arithmetic Test to 143 pupils in grade four and 136 in grade six. In grade four the understanding test scores correlated .665 and .751 with problem-solving and computation. In grade six the corresponding R's were .751 and .756.

The writer (in 1960) prepared tests of mathematical understanding for grades three through six with reliability coefficients of .893 to .897.[1] Since results of these tests will be discussed later, only two test samplings (fourth and sixth) will be related to the problem being presented at this point. In a sampling of 433 fourth-grade children, the tests of understanding UCCT correlated .637 with scores on the Reasoning Test section of the California Achievement Test CAT and .499 on the CAT Fundamentals Tests. For a sampling of 495 sixth-grade pupils comparing UCCT with the CAT, the results were similar to the fourth-grade test and very close to the findings of Spainhour. The correlation between UCCT and CAT Reasoning Test was .657 and .691 between UCCT and CAT Fundamentals Test.

Careful study of most standardized achievement tests in arithmetic, containing subtests on problem-solving or reasoning, reveal the inadequacies of these tests to measure pupil understanding of arithmetical processes.

Other methods for studying pupil understanding of arithmetic

Methods of studying pupils' thinking in arithmetic were carefully discussed by Guy Buswell (16) in 1949. He suggested six ways to study pupils' thinking (have pupil think aloud, use manipulative aids, make diagrams, and the like) as the basis for understanding how pupils arrive at their answers in computation or in solving word problems. He concluded with the thought:

> . . . the problem of the teacher is to understand the successive levels of pupils' thinking and to help children develop clear

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[1] The University of California Arithmetic Comprehension Tests will be designated as UCCT. California Achievement Tests will be referred to as CAT.

> meanings at every stage of the learning process. These meanings exist in the pupils' thinking. Arithmetic teachers must learn to understand how pupils think (16).

Vincent J. Glennon did a frontier type of research study on the testing of meanings in arithmetic in 1949. He reported six causes for the lag in the development of adequate methods for measuring pupil growth in understandings and meanings in arithmetic: (1) the change in the objectives in the teaching of arithmetic to include meaningful teaching; (2) the impact of physiological psychology on methods of teaching; (3) the security teachers and supervisors found in present practices based on drill and testing; (4) the lag in the development of evaluative instruments; (5) the dependence of teachers upon standardized tests and teaching for these tests; and (6) the lack of a definite list of understandings and meanings. Glennon pointed out the paucity of the research studies in the area of testing for meanings and concluded that this was one of the most neglected educational problems of the day (39). He conducted a study to discover the extent of growth and mastery of certain basic mathematical understandings possessed by representative groups on seven educational levels. A sampling of arithmetical understanding (measured by an objective-type test of 90 items covering concepts taught in grades one through six) was taken in grades seven, eight, nine, twelve, college freshmen, college seniors, and teachers in service.

Glennon concluded that the average seventh-grade pupil tested in his study had acquired 12.5 per cent of the understandings basic to the computational processes taught in grades one through six. The scores increased with the grade level to 14.01 per cent for eighth grade; 18.02 per cent for ninth grade; and 37.00 for twelfth grade. These data present evidence of the meager degree with which teachers were succeeding in helping children to understand basic arithmetical concepts measured by this one test. Certainly, in 1949, teachers were not making meaningful teaching one of their primary objectives in elementary arithmetic work.

J. Fred Weaver, in 1955, followed the suggestions made by Buswell for studying children's thinking patterns in arithmetic and planned an investigation involving interviews in which children individually "think out loud" in response to specific quantitative problems. The technique was illustrated in actual operation by a fourth-grade teacher in the teaching of multiplication combinations. Pupil responses and thought patterns were recorded in the form of a class chart to facilitate study of the data. Class organization and instructional practices were then changed to help individual pupils learn and understand the multiplication combinations. The teacher then used the interview technique to gain helpful information in planning her instruction throughout the year.

The conclusions drawn by Weaver on this one evaluative technique were:

> . . . every teacher can make a sound investment by interviewing children in her class periodically to determine and study their levels of thinking when dealing with various quantitative situations. By then using the knowledge gained from such interviews to assist her in providing a helpfully differentiated program of teaching and learning experiences, every teacher can reap big dividends in the form of increased instructional effectiveness (89).

Another method for sampling pupil understanding of the numeration system was used by Frances Flournoy. Basic concepts for understanding decimal numeration were stated, and four tests on different levels of difficulty were designed to test pupil understanding of these selected principles of numeration. The tests were administered in grades four through seven. The results of the test given in grade seven were representative of the findings of the study: on the twenty-five item numeration system understanding test administered at midyear, these

106 seventh graders ranged from 6 to 23 items correct with a mean of 15.12 items correct. The mean per cent correct was found to be 60.48 (35). The researcher concluded by saying that while the classroom arithmetic exercises related to the reading and writing of numerals were important, they did not sufficiently explore and deepen pupil understanding of the numeration system.

Rappaport made a similar study, in 1958, to test the extent of understanding meanings in arithmetic by seventh- and eighth-grade pupils. Assuming that a score below 50 per cent was an indication of inadequacy, he discovered that pupils did not have an adequate understanding of the meanings in arithmetic. Computational skill was not an indication of the understanding of the meanings of the processes used in computation, for some pupils with very high computational scores achieved very low scores on the meaning test. Pupils showed a greater concern over the securing of answers to arithmetic exercises than in understanding the meanings in arithmetic. Rappaport stated: "The results of this study indicate a wide discrepancy between computational skill and understanding. Teachers and administrators should make every effort to narrow this gap between skill and understanding (62)."

Teachers' and prospective teachers' understanding of arithmetic

A study of university students' comprehension of arithmetical concepts was carried out by the writer in 1961. This study was designed to measure students' understanding of arithmetical concepts as they progressed through courses designed to teach these processes. Two classes (fifty-five prospective elementary school teachers) were tested with a sixth-grade arithmetic comprehension test before and after completing a lower division mathematics course for elementary school teachers.

The conclusions drawn were that prospective elementary school teachers adhered to many arithmetical concepts and procedures learned in elementary and junior high school. While there were many basic arithmetical concepts understood by these prospective teachers, the students still clung to traditional methods and mechanical procedures when attempting to explain the following concepts: (1) partial products in multiplication; (2) placement of quotient figures in long division; (3) placement of the decimal point in problems involving decimal fractions; and (4) understanding of and use of denominate numerals. Future gains in the education of prospective teachers of arithmetic must be based upon adequate analysis of student misconceptions and systematic efforts to assist each student to build new concepts based upon his present knowledge (30).

Cheney (20) constructed a test using many test items from the instrument prepared by Dutton to measure elementary school teachers' understanding of basic arithmetical concepts. The test was administered to 120 teachers in eight selected elementary schools in Ventura and Los Angeles counties. His conclusions were similar to the findings of Dutton, that teachers understand some aspects of arithmetic they teach to children but have difficulty with other phases. The most difficult items for teachers (in this one sampling) were place value, remainders in division, fractions involving multiplying and dividing, comparison of decimal fractions, and use of denominate numbers. Primary-grade teachers (K-3) scored much lower than intermediate-grade teachers—the mean scores being 29.91 for primary- and 38.74 for intermediate-grade teachers out of a possible score of 52. This difference was significant well beyond the one per cent level.

Cheney's study is important for several reasons: (1) His careful review of research studies dealing with the evaluation of teachers' understanding of arithmetic points out the paucity of studies and valid instruments in this area. (2) He accentuates the important fact that teachers have difficulties with certain aspects of arithmetic and that wide differences exist among teachers. (3) Primary-grade teachers seem to have an understanding of the simple concepts taught to young children and have neglected more advanced concepts taught in intermediate grades.

The few available research studies indicate that elementary school teachers should examine their own understanding of the basic meanings in arithmetic. Having learned their arithmetic in a mechanical way and often as a skill subject, they may have serious difficulties teaching arithmetic meaningfully to children. Thus, meaningful teaching and meaningful learning are closely related. The evaluation of pupil understanding of arithmetical concepts cannot be divorced from evaluation of teacher understandings of the same processes.

Evaluation of pupil understanding of arithmetic in Europe. One study made in the French-speaking countries of Europe has a direct connection with the importance and evaluation of pupil understandings of arithmetical concepts. A team of educational psychologists in Paris interviewed children between the ages of eleven and thirteen to discover their difficulties in working arithmetical problems. The most common mistakes were caused, according to this study, by failure to understand the problems from a practical point of view and the incapacity to show the correct mathematical relationships corresponding to the elements involved in the problems. Very few children were able to correct a mistake made in their own problems (44).

Arithmetic in English and Scottish schools

William Brownell has been studying arithmetic teaching in English and Scottish schools. While his studies are not at this time directed toward pupil understanding or arithmetical processes, he has been able through the use of observation techniques to point out certain characteristics of arithmetic instruction in lower grades. Brownell believes:

> . . . we have seriously underestimated the attention span of school beginners. We have held that beginners cannot remain interested in anything and keep their attention thereon for more than a relatively few minutes at a time—say fifteen, twenty, or thirty minutes. In the English and Scottish schools we saw infant school children working happily, busily, and effectively at number tasks throughout periods of an hour or more. We have seriously underrated the "readiness" of school beginners for systematic work in arithmetic. We can safely ask children in the lower grades to learn much more in arithmetic than we are now asking them to learn (8).

Guy Buswell (15) made a comparative study of achievement in arithmetic in England and Central California. The results of this study indicate that English children, who begin formal arithmetic at five years of age and vigorously prepare for their "11-year plus" examinations, achieve much more than California pupils of the same age. This study does not deal with pupil understanding of arithmetical concepts. It does, however, point out the importance of the philosophy and objectives of education. If we so desired, arithmetic could be introduced earlier, time allotments could be increased, and additional content could be taught. Evaluation of such a program should then be made in terms of achievement and understanding.

Buswell's study does not indicate a lack of ability to learn arithmetic on the part of California pupils but rather that the school programs in England and California are quite different. Also the place of meaningful teaching and the evaluation of pupil understandings have again been overlooked in an important comparative study! Neither English nor American school systems should be satisfied with an arithmetic program based upon computational skills alone.

CONCLUSIONS ON EVALUATION OF PUPIL UNDERSTANDING OF ARITHMETIC

The evaluation process involves three important elements: (1) evaluation of the comprehensive range of objectives, (2) the use of a variety of techniques to secure data on pupil progress, and (3) emphasis upon the child's total personality development. The application of modern evaluation processes to the arithmetic program has been disappointingly slow and continues to be the most neglected aspect of arithmetic teaching.

Standardized achievement tests have not been designed to measure all arithmetic outcomes. They are not valid measures of the classroom teacher's instructional objectives, and they place undue emphasis upon computational skill. At best these tests provide rough estimates of pupil accomplishment in arithmetic. Regardless of the shortcomings of these tests, they have been used to show teaching proficiency, pupil accomplishment, and even effectiveness of some modern mathematical programs! The strangle hold of this type of measurement must be broken in order to establish procedures which are valid and which cover the entire range of the teaching-learning act.

Understanding the successive levels of pupils' thinking and helping children develop clear meanings at every stage of the learning process are basic to sound instruction in elementary school mathematics. In fact, the extent to which provision is made for these factors will determine the success or failure of modern mathematical programs.

Research studies based upon a variety of tests show that teachers and administrators have not made meaningful teaching one of the primary objectives in the elementary school arithmetic program. Objectives for elementary school mathematics have been clearly stated, and adequate techniques for evaluation have been devised; thus, the main excuses for the faltering delays in the application of modern evaluation procedures have been eliminated. The challenge of introducing modern mathematics programs and insuring their success will be thwarted if present evaluation practices are continued.

Evidence is available to show that there is a very close relationship between meaningful teaching and meaningful learning. The classroom teacher must have a clear understanding of each new arithmetical concept presented to children. The verbal language and instructional aids used must likewise be meaningful to pupils and adjusted to their individual differences.

Studies made of arithmetic programs in other countries provide valuable information pertaining to readiness for learning, length of instruction time, use of sensory-perceptual aids, methods of teaching, and importance of educational philosophy. Stress needs to be placed upon achievements in teaching and in learning which enable pupils to make optimum growth in mathematics rather than upon differences between national programs.

NOTES AND PROBLEMS

1. Secure a copy of a standardized achievement test used in elementary schools near your home, or where you work if you are a teacher or administrator. Study each test item carefully to see if pupil understanding of mathematical processes are being measured.
2. Why has there been such a dearth of tests to measure pupil understanding of arithmetic?
3. Children's thought process, as they work on mathematical problems, should be developed and directed. Several different thought processes may be used by children to work a problem in subtraction with whole numbers. To test this last statement, ask several children to say aloud the actual thinking used to secure an answer for the example $421 - 232$.

4. How well do you understand basic arithmetical concepts taught in the elementary school? Turn to the appendix and take the sixth-grade comprehension test. Make a list of concepts and processes which you do not understand or which were not taught when you were in sixth grade.
5. Read Guy Buswell's comparative study (15), "Achievement in Arithmetic in England and Central California." Write a critical analysis of the article. Are comparisons adequate and valid?

IV

Practical Applications of Evaluative Techniques

In Chapter III the necessity for evaluating pupil understanding of arithmetical concepts was discussed and pertinent research studies were reviewed. One of the important conclusions made was that little difference exists between teaching for understanding and testing of understanding. The teacher must be aware of the language and aids used to guide children's thinking and understanding involved in learning each new concept. This type of teaching requires the use of a variety of evaluative procedures for discovering pupil growth in understanding new arithmetical concepts. This chapter will include: (1) the construction and use of tests to measure pupil understanding of selected arithmetical concepts in grades three through six and (2) suggestions for informal evaluation which should be used in the day-by-day teaching of arithmetic.[1]

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[1] Sample Arithmetic Comprehension Tests (grades three through six) have been included in the appendix.

TESTS FOR MEASURING PUPIL UNDERSTANDINGS

There are few valid and reliable tests available for teacher use in evaluating pupil understanding of arithmetical meanings. While textbook publishers are including evaluation tests in their revised editions prepared because of the mathematical revolution, teachers and curriculum workers still need to prepare their own tests to measure this crucial aspect of the arithmetic program in the elementary schools. There are distinct advantages in teacher-made tests because:

1. Each school system should make its own tests to cover particular instructional objectives.
2. Tests can reflect new content, changes in methodology, and local standards.
3. When teachers prepare test items, they must thoroughly understand the concepts being measured.
4. Improvement of instructional practices and curriculum development should be based upon valid evaluation of the objectives established for the total program.
5. More opportunity exists for involvement of the professional education staff and lay workers in this type of evaluation.

The sample tests which will be presented in this chapter may be used as beginning instruments for evaluating selected arithmetical concepts in grades three through six. It is hoped that these tests will inspire teachers to check pupil understanding of each new arithmetical concept they teach. Experience with using these test items should encourage teachers to develop other test items and evaluative techniques especially suited to the children and the concepts being taught.

Suggestions for test construction

Each of the tests reported in this chapter has been developed according to sound principles of test construction. The reliability coefficients are within the range of acceptability established by authorities (.80 to .95) (40). The validity has been carefully provided for by content analysis of books and courses of study in grades three through six and the use of a panel of teachers and supervisors to determine the curricular validity of concepts included in the test.

Specifications for Tests. A study was made to determine the major arithmetical concepts presented at each grade level. This was accomplished with the assistance of experienced, elementary school teachers, by searching through the major arithmetic textbooks and courses of study and by asking a select group of college teachers of mathematics for their assistance and appraisal of the items to be covered. The writer is well aware of the fact that grade-level standards are very artificial. However, the tests do not have to be used in their entirety. Selected items may be used at appropriate times after a concept has been taught. Teachers should be encouraged to use appropriate evaluative techniques for each stage of the sequential development of arithmetic concepts—from beginning perceptual learnings to use of abstract symbols.

Objective-type, multiple-choice test items were prepared because of the ease and objectivity of scoring. Approximately two times as many test items were prepared as were finally included in each sample test. A variety of test items was used so that a good sampling of concepts could be made and so that each test would be interesting as well as challenging to pupils. Some of the choices for incorrect responses were made by presenting children with the main "stem" or question and asking them to give their answers. Elementary school teachers were especially helpful in suggesting possible answers for multiple-choice test items because they were aware of misconceptions and incorrect responses of children in the areas being tested.

Tryout of Test Items. Preliminary test forms were prepared and used with approximately 150 children at each grade level. Revisions were made, and the tests were administered in several cities in Southern California; Harford County, Maryland; and Texas. In this second testing approximately 500 children at each grade level were tested. Additional revisions have been made, and the tests have been used with over 1,000 pupils at each grade level.

Norms for the Tests. Since these tests are to be used as diagnostic tests, no norms have been established. Studies of the difficulty of each test item have been made in order to check reliability of the tests and to make appropriate revisions. Theoretically, one would expect children to make high scores on the tests if instruction was adequate and if the new concept being taught was within the ability level of the pupils involved. Interpreting and using of test results will be made in connection with the discussion of each test.

Arithmetic comprehension test, grade three

The third-grade test will be presented by using the results obtained from one application of the test to 256 pupils in a city school system located in Texas. Schools used in the study were selected on the basis of socio-economic status. One school represented a relatively high socio-economic level, three schools enrolled pupils from a middle-class district, and two schools had pupils from lower socio-economic levels. There were 120 boys and 136 girls. The I.Q. range was from 90 to 137 with a median of 104. The testing was given during the first week in May, 1961, so that the basic arithmetical concepts for the third grade would have been completed. The results are shown in Table I.

TABLE I

Analysis of Pupil Difficulties in Comprehension of Third-Grade Arithmetic

Item	*Description*	*Incorrect responses*	*Per cent of total:*	
			wrong	*right*
1	Place value—tens	15	6.0	94.0
2	Place value—tens	14	5.0	95.0
3	Place value—hundreds	12	5.0	95.0
4	Use of zero	72	28.0	72.0
5	Counting	34	13.0	87.0
6	Counting	20	8.0	92.0
7	Understanding addition	22	8.0	92.0
8	Addition with three-place numbers	28	11.0	89.0
9	Understanding addition	12	5.0	95.0
10	Addition vocabulary	140	55.0	45.0
11	Column addition	34	13.0	87.0
12	Adding one and two places	67	26.0	74.0
13	Place value—tens	156	60.0	40.0
14	Understanding subtraction	36	14.0	86.0
15	Understanding subtraction	76	30.0	70.0
16	Regrouping in subtraction	50	20.0	80.0
17	Understanding subtraction	12	5.0	95.0
18	Understanding subtraction	46	18.0	82.0
19	Regrouping in subtraction	156	60.0	40.0
20	Understanding subtraction	58	23.0	77.0
21	Fourths—fractional parts	84	33.0	67.0
22	Thirds—fractional parts	62	24.0	76.0
23	Halves—fractional parts	29	11.0	89.0
24	Fourths—fractional parts	127	50.0	50.0
25	Telling time	12	5.0	95.0
26	Telling time	41	16.0	84.0
27	Triangle	14	5.0	95.0
28	Liquid measure	63	24.0	76.0
29	Linear measure	22	8.0	92.0
30	Roman numerals	34	13.0	87.0
31	Roman numerals	55	21.0	79.0
32	Seasons of the year	132	52.0	48.0
33	Money—signs and symbols	16	6.0	94.0
34	Money—addition	29	11.0	89.0

There are three ways to analyze the test results for grade three: city-wide results, individual school performance, individual pupil performance.

City-wide considerations. The two most conspicuous difficulties which should be noted by curriculum workers and supervisors interested in city-wide performance on this test are items 13 and 19:

13. In the problem $\begin{array}{r}14\\+18\\\hline\end{array}$, what number is carried?
    a. 1 one
    b. 2 ones
    c. 1 ten
    d. none of these

19. $\begin{array}{r}48\\-19\\\hline\end{array}$ To subtract in this problem, we must:
    a. carry
    b. regroup the 4 tens
    c. regroup the 1 ten
    d. subtract without doing anything else first

These two test items involve carrying (one ten) and regrouping (four tens). There is a good possibility that these two processes have been taught mechanically or that different terminology has been used. The fact that 60 per cent of the pupils selected incorrect responses is significant, particularly when these are key concepts in third-grade arithmetic.

Although reading difficulties were reduced to a minimum and test administrators were requested to read test items where pupils had difficulties (except key concepts or ideas), item 10 may involve a word clue which was not taught or understood by children in some classrooms:

9. Three girls were playing ball. Two boys came to join them. How many children were there altogether. Would you:

a. subtract
b. add
c. multiply

10. The clue to the problem above is:
a. playing ball
b. how many
c. altogether

Although teaching children to look for clues in problem-solving is not too helpful, the identification of obvious helping words is one way to give children assistance.

Seasons of the year, item 32, was missed by 52 per cent of the children:

32. November comes in which season of the year?
a. spring
b. autumn
c. winter
d. summer

This could mean that little attention was given to this topic or pupils did not know the word *autumn*. They may have been taught *fall* instead of *autumn*.

Other test items which reveal useful information for city-wide study are:

4. In the number 90, the zero:
a. doesn't have to be there
b. means nothing
c. holds the ones place
d. holds the tens place

This problem involves the use of zero as a place holder. If 28 per cent of the students did not know this important fact, some study should be made to discover the difficulties involved in teaching the process.

In item 15, 30 per cent of the pupils did not select *d* as the correct response:

15. 87 To subtract in this problem, we must:
$$\begin{array}{r} 87 \\ -15 \\ \hline \end{array}$$

a. make the 7 ones larger
b. make the 1 ten larger
c. make the 5 ones larger
d. none of these

Other test items which showed pupil learning problems were: identifying the colored part of a circle (item 21), missed by 33 per cent; size of a quart (item 28), difficult for 24 per cent; and reading the Roman numeral XII (item 31), missed by 21 per cent of the pupils.

Individual School Performance. The results of this test should be studied by each school involved in the sampling. One school might be especially weak in the meaningful teaching of some particular concept. This fact might also show up on the city-wide test results because of the large number of pupils missing a test item, such as number 19, which deals with regrouping in subtraction.

Individual Pupil Performance. The use of test results to study individual pupil progress in understanding arithmetic concepts is probably the most important of all three interpretations. The classroom teacher should discover pupil weaknesses and remove them if possible by the close of the school term or let the next year's teacher know about the inadequacies. This, of course, raises the important question of when to test. Obviously, the testing must be an integral part of every new learning situation. This test was given at the close of the year primarily for standardizing the test and for discovering general teaching procedures in one school system.

Arithmetic comprehension test, grade four

The fourth-grade test was administered to 212 children in one school system in Ventura County, California. The testing was done during the last of May, near the close of school. The children were selected from all elementary schools in the district and covered all socio-economic levels. The I.Q. range was from 86 to 145. The range of scores, item analysis, and interpretation of data follow.

The histogram (see Fig. 4-1) shows the scores made by fourth-grade pupils. The range was 8 to 40, with a possible score of 48. The mean score was 23.56, the median 25.56, and the mode 33.

The distribution was approximately normal. Both the mean and the median fell near a score of 50 per cent on the test, indicating that the average student understood about half of the concepts included in the test.

Item Analysis. The item analysis for the fourth-grade stu-

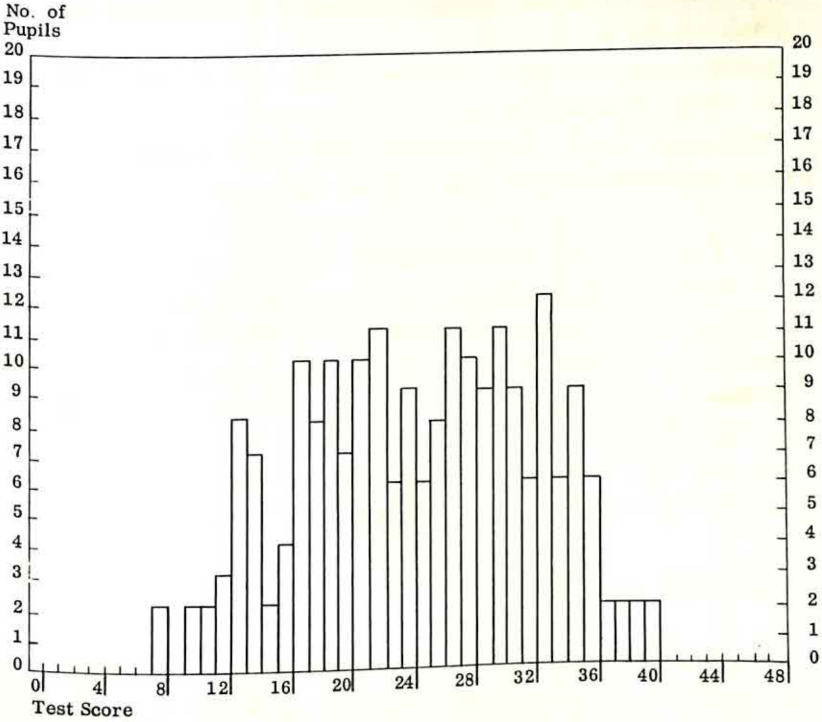


Fig. 4-1. Distribution of fourth-grade pupils' scores on arithmetic comprehension test.

dents' scores is shown in Table II. The scores were arranged in numerical order; the upper 25 per cent make up the *upper quartile;* the lowest 25 per cent make up the *lower quartile.*

The *discrimination index* indicates the degree to which each item discriminates between the upper quartile and the lower quartile. The higher the discrimination index, the greater the discrimination between the two groups. A negative discrimination index (items 16, 30, 31, and 46) shows that the lower quartile gave more correct scores on the item than did the upper quartile; hence, a negative discrimination.

The *difficulty index* indicates how difficult each item was for the entire group taking the test. The higher the difficulty index, the easier the item.

In general, an item is thought to be a good test item for group testing if the discrimination index is 20 or above and the difficulty index is 39 or above.

Interpretation of Item Analysis. The ten test items which were least difficult for the upper quartile and the lower quartile and the ten items which were most difficult for the two groups are shown in Fig. 4-2, on page 72.

TABLE II

Item Analysis of Fourth-Grade Pupils' Scores on Arithmetic Comprehension Test

Item	*Upper quartile*	*Lower quartile*	*Discrimination index*	*Difficulty index*
1	84	28	57	56
2	82	24	58	53
3	88	44	49	66
4	94	40	62	67
5	96	58	56	77
6	76	28	48	52
7	92	64	40	78
8	92	72	32	82
9	72	26	46	49
10	72	40	33	56

Item	*Upper quartile*	*Lower quartile*	*Discrimination index*	*Difficulty index*
11	100	66	65	82
12	94	60	48	77
13	32	10	32	21
14	100	44	75	72
15	90	38	57	64
16	26	32	–7	29
17	96	36	69	66
18	82	12	69	47
19	96	58	56	77
20	80	16	63	48
21	70	12	60	41
22	90	20	70	55
23	92	32	64	62
24	100	28	82	64
25	78	28	50	53
26	74	26	48	50
27	56	18	41	37
28	90	48	50	69
29	70	34	37	52
30	14	16	–3	15
31	24	26	–3	25
32	94	16	77	55
33	50	16	39	33
34	96	28	73	62
35	44	26	20	35
36	58	14	48	36
37	94	24	72	59
38	98	44	71	71
39	38	16	28	27
40	54	10	51	32
41	74	36	39	55
42	66	18	49	42
43	28	18	13	23
44	50	16	39	33
45	36	26	12	31
46	4	16	–30	10
47	52	20	35	26
48	82	40	45	61

UPPER QUARTILE			LOWER QUARTILE	
Item No.	% Giving Correct Answer		Item No.	% Giving Correct Answer
14	100	Highest Ten Items	8	72
11	100		11	66
24	100		7	64
38	98		12	60
5	96		5	58
17	96		19	58
19	96		28	48
19	96		3	44
34	96		14	44
4	94		38	44
12	94			
32	94			
37	94			
33	50	Lowest Ten Items	20	16
35	44		30	16
39	38		32	16
45	36		39	16
13	32		44	16
43	28		46	16
16	26		36	14
31	24		18	12
30	14		21	12
46	4		13	10
			40	10

Fig. 4-2. Test items least difficult for upper quartile and items most difficult for both groups.

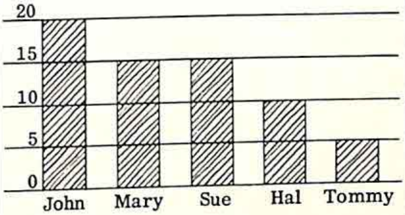
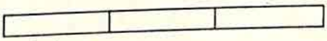
Item No.	% Top Group	% Low Group	Disc. Index	Diff. Index	
14	100	44	75	72	Temperature is measured in a) pounds b) degrees c) hours d) days
11	100	66	65	82	The longest time is a) month b) year c) week d) second
38	98	44	71	71	Look at the example  Who got the most words right? a) John b) Mary c) Hal d) Tommy
5	96	58	56	77	The most money is a) $1.40 b) $4.10 c) $4.01 d) $0.41
19	96	58	56	77	This line is cut into  a) halves b) fifths c) fourths d) thirds

Fig. 4-3. Test items easiest for both groups.

Although the concept of perimeter is presented in the fourth grade, several other concepts also must be understood in order to answer this question: distance, surface, total.

The difficulty the children seemed to have with item 39 may be partly due to the form in which the graph was drawn. If the numbers had appeared at the end of each line rather than above each one, there might have been less chance for error in reading the graph.

The fact that a multiplication and a division concept appear in this difficult group may indicate that the fundamental processes are being taught more by rote and less by understanding than one might hope.

Arithmetic comprehension test, grade five

The fifth-grade comprehension test was prepared according to the principles listed earlier in the chapter and two revisions were made before administering it to a sampling of 357 B6 pupils. Owing to midyear promotional practices, pupils enrolled in B6 classes were chosen for this study because they had just completed the fifth-grade work. Four elementary school districts in the Los Angeles area participated in this study. Children were selected from upper middle-class, average middle-class, and lower middle-class schools. The I.Q. range was from 80 to 147, measured by the California Mental Maturity Test. There were 176 boys and 181 girls.

The results of this study will be shown in terms of the percentage of pupils selecting the correct answer for each concept measured by the test. Read Table III from left to right —test item, per cent of pupils getting correct answers, and the concept measured.

The results given in Table III show clearly that many arithmetical concepts have not been learned by fifth-graders involved in this study. There were 25 test items where pupil scores were below 60 per cent understanding. Certain key

The five test items which were easiest for both groups are shown in Fig. 4-3. The concepts covered by these items are related to temperature, time, graphs, money, and fractional parts. This would indicate that measurement of various kinds is better understood by these fourth graders than other arithmetical concepts in the test.

The four test items which proved to be most difficult are shown in Fig. 4-4. The concepts involved are multiplication, division, perimeter, and interpretation of a graph.

Item No.	% Top Group	% Top Group	Disc. Index	Diff. Index	
46	4	16	-30	10	In the following problem 480 we think x 5 a) 5 x 8 tens are 40 tens, carry 0 b) 5 x 8 tens are 40 ones, carry 4 tens c) 5 x 8 tens are 40 hundreds, carry 4 hundreds d) 5 x 8 tens are 40 tens, carry 4 hundreds
30	14	16	-3	15	When dividing the problem 4/16 we think a) divide 1 by four b) 1 ten will not divide into 4 equal groups of ten c) what number subtracted from 16 equals 4 d) sixteen is to be multiplied by 4
13	32	10	32	21	Perimeter means a) distance across any surface b) surface of anything c) distance around the outside of any surface d) the total distance of two surfaces
39	38	16	28	27	(Referring to same graph as in item 38 on previous page) How many words did Mary get right? a) 10 b) 20 c) 15 d) 5

Fig. 4-4. Four test items most difficult for both groups.

concepts, place value (items 15, 16), concepts of measurement (items 35, 39, 42), and knowing the base of our number system (item 21), indicate that for most children in this study arithmetic had been taught in a mechanical way with heavy emphasis upon computation rather than understanding.

While more detailed analysis could be made of the data, the main purpose of showing how the fifth-grade arithmetic comprehension test could be used in diagnostic work with school systems or a particular class has been achieved. A sample of the test used has been included in the appendix.

TABLE III

Per Cent of Correct Pupil Responses on the Fifth-Grade Arithmetic Comprehension Test

Item	*Per cent correct*	*Arithmetical concept measure*
16	1.0	Regrouping in division—place value
15	17.0	Place value
42	20.0	Measuring devices and terms
39	22.0	Square measure
35	24.0	Reading a line graph
32	26.0	Decimal equivalents
21	29.0	Base of number system
36	31.0	Area
45	31.0	Denominate numbers—regrouping
19	34.0	Multiplication factors
11	35.0	Addition of fractions
37	37.0	Perimeter
43	41.0	Denominate numbers—addition
5	42.0	Meaning of mixed numbers
41	43.0	Measuring device
40	48.0	Area
38	48.0	Perimeter
29	49.0	Decimal meaning
10	49.0	Reduction of fractions
1	51.0	Rounding off

Item	*Per cent correct*	*Arithmetical concept measure*
14	52.0	Place value
25	55.0	Place value in partial product
13	55.0	Addition of fractions
28	57.0	Measuring device
26	57.0	Division—quotient place value
30	64.0	Decimal place value
17	64.0	Roman numerals
8	68.0	Multiplication—problem-solving
18	70.0	Roman numerals
31	71.0	Fractional units of measure
6	72.0	Fractional units of measure
4	76.0	Fractional part of a number
9	77.0	Subtraction of fractions
7	78.0	Division—problem-solving
12	79.0	Picturing fractional meaning
34	80.0	Interpretation of bar graph
23	81.0	Multiplication—problem-solving
24	81.0	Division—problem-solving
3	81.0	Decimal comparison
2	83.0	Decimal comparison
44	81.0	Measurement term
27	84.0	Interpreting circle graph
33	87.0	Interpreting bar graph
22	89.0	Subtraction—problem-solving
20	91.0	Value of money

Arithmetic comprehension test, grade six

The sixth-grade arithmetic comprehension test was administered to 495 pupils in the Los Angeles area. Schools were selected so that a cross section of the city's population was obtained. Tests were given during the final month of school so that scores from standardized achievement tests would be available and so that pupils would have covered the main arithmetical concepts required at this grade level. Comparisons were made between pupil performance on the California

Achievement Test in Arithmetic and the Arithmetic Comprehension Test developed at U.C.L.A.

I.Q. scores for this study ranged from 70 to 155 with a median of 107.7. The results of the Achievement Test are shown in Table IV. The median for CAT computation was 7.39. For arithmetic reasoning on the CAT the median was 7.83. Considering the I.Q. distribution for this sampling, these achievement records are quite good.

TABLE IV

Sixth-Grade Arithmetic California Achievement Test Results and I.Q. Scores

Grade level	*CAT computation*	*CAT reasoning*	*I.Q. score*	*Pupils*
			155	1
11.5	1		150	1
11.0	1	1	145	1
10.5	8	1	140	2
10.0	10	3	135	8
9.5	32	20	130	15
9.0	46	51	125	18
8.5	56	59	120	48
			115	53
8.0	80	80	110	66
7.5	62	39	105	74
7.0	70	63	100	64
6.5	62	89	95	53
6.0	42	32	90	46
5.5	19	17	85	23
5.0	4	22	80	9
4.5	0	8	75	10
4.0	2	5	70	3
3.5	—	3		
3.0	—	2		
2.5	—	—		
2.0	—	—		
1.5	—	—		
1.0	—	—		

N = 495
M = 107.7

N = 495 M = 7.39 M = 7.83

Pupil performance on the arithmetic comprehension test is shown in the histogram, Fig. 4-5. The range was 2 to 40, with a possible score of 46. The median was 20.93. The distribution of scores is skewed to the lower end of the scale, showing very little understanding of the arithmetic concepts measured by this test. The fact that the central tendency is well below 50 per cent correct on the comprehension test is indicative of

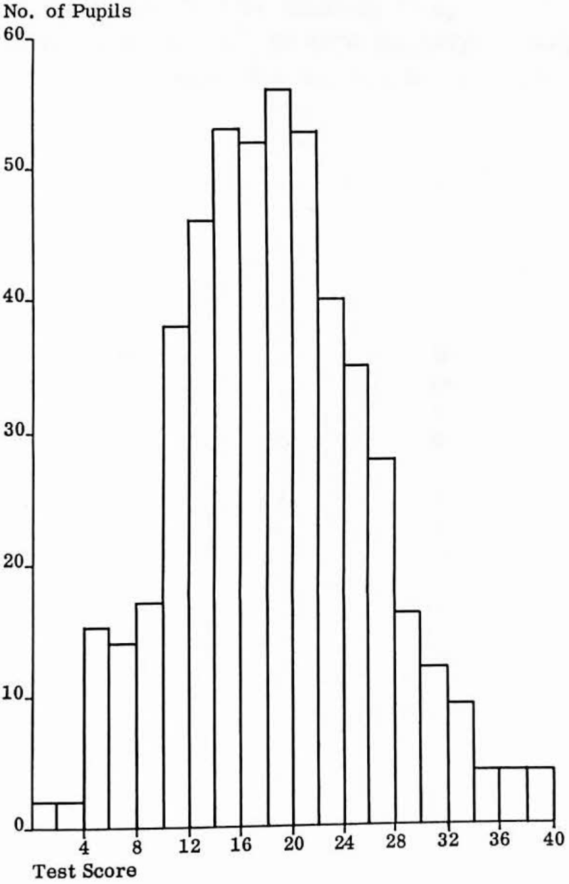


Fig. 4-5. Sixth-grade pupil performance on University of California Arithmetic Comprehension Test.

poor pupil performance. Concepts which were most difficult for pupils to understand were: place value in quotient in division; understanding multiplication and division; understandng multiplication and division of fractions; work with denominate numbers; partial products in multiplication; and placement of the decimal in decimal fractions.

The discrimination and difficulty index is shown in Table V. Reading the table from left to right: on the first test item, 94 per cent of the upper quartile and 52 per cent of the lower quartile gave correct answers to the test item; the discrimination index was 55 and the difficulty index was 73.

TABLE V

Discrimination and Difficulty Index for UCCT Arithmetic Test, Grade Six

Item	*Upper quartile*	*Lower quartile*	*Discrimination index*	*Difficulty index*
1	94	52	55	73
2	96	58	56	77
3	82	12	69	47
4	86	12	72	49
5	80	36	46	58
6	74	10	65	42
7	70	4	72	37
8	56	16	44	36
9	28	18	13	23
10	38	22	–4	40
11	90	44	53	67
12	80	2	81	41
13	28	0	70	13
14	36	22	17	19
15	56	10	53	33
16	72	58	15	65
17	78	4	76	41
18	84	8	75	46
19	8	10	–5	9
20	66	24	43	45

Item	*Upper quartile*	*Lower quartile*	*Discrimination index*	*Difficulty index*
21	64	10	58	37
22	46	18	32	32
23	82	28	54	55
24	90	30	63	60
25	82	32	51	57
26	44	24	22	34
27	72	42	31	57
28	92	42	58	67
29	48	8	51	28
30	96	22	76	59
31	48	14	40	31
32	68	12	58	40
33	68	14	56	41
34	68	6	67	37
35	92	48	53	70
36	22	6	31	14
37	22	10	21	16
38	12	18	11	15
39	50	8	52	29
40	82	44	41	63
41	62	20	44	41
42	88	20	67	54
43	60	2	73	31
44	60	16	47	38
45	44	12	39	28
46	80	30	51	55

CONCLUSIONS ON MEASUREMENT OF PUPIL UNDERSTANDINGS

The results of the UCCT tests administered to pupils enrolled in grades three through six in four different school systems pointed out the inescapable fact that mathematical understandings are not being adequately taught or evaluated in these elementary schools. While lip service has been given to the importance of teaching arithmetic in a meaningful way, little has been done to accept the point of view that as new processes are presented the basic understandings must be developed and evaluated. Because of this discrepancy between teaching and evaluation of all objectives of arithmetic instruction, the preparation of practical, day-by-day procedures for evaluation have been presented in the next section.

INFORMAL EVALUATION OF PUPIL UNDERSTANDING OF ARITHMETICAL CONCEPTS

Making use of available evaluation techniques and learning how to develop new methods of evaluation are not easy for the elementary school teacher. Good judgment must be used in deciding how much evaluation will be used by a particular teacher with a particular group of children. Since evaluation activities require careful planning and continued emphasis if they are to be successful, suggestions for beginning evaluative work are needed:

1. The teacher needs to know the specific objectives established for instruction. Do not attempt too much. Each lesson should be centered around some specific purpose.
2. The teacher must know the kind of behavior which indicates pupil growth toward the objectives. If the teacher cannot tell the difference between a child who understands a new process and one who does not, the objective is inadequate or the teacher does not know how to present the new step meaningfully. This ability to measure pupil growth toward achieving an objective constitutes one important way to test the value and appropriateness of an objective.
3. Next, the teacher must collect valid evidence of pupil growth toward achieving the objectives. As has been indicated, much of this can be done as the lesson is taught and closed. The techniques used might include: (1) observing pupils as they work sample problems, (2) asking a pupil to demonstrate the problem with a drawing or by using an instructional aid, (3) encouraging the pupils to make their own problems using the new arithmetical process, and (4) giving a short paper and pencil test which covers computational skills, understanding of the concept being taught, and application of the new learning.
4. The next step is to make use of the evidence collected. Again note that through observation and work with pupils, the desired behavior

is recognized and secured. Thus, elaborate records of pupil growth may not be needed. However, if some pupils have mastered the new arithmetical step or concept and others have not, notes must be made to direct additional planning and instruction. If some provision is made for grouping and individualized instruction, records must be kept of pupil progress to insure continuity of teaching and learning.

5. Finally, the teacher should concentrate on evaluation which involves diagnosing of pupil difficulties, determining pupil readiness for new processes, and appraising pupil understanding of basic mathematical concepts. While the long-term aspects of evaluation involving achievement and survey comparisons are important, the classroom teacher should leave much of this work to curriculum workers and administrators and concentrate on the day-by-day instructional program.

Since this book is concerned primarily with the evaluation of mathematical understanding, the illustrations which follow will deal with procedures which can be applied in directing pupil growth in the understanding of some of the most crucial mathematical concepts which children must learn as they advance through the elementary school.

Practical evaluation at primary level

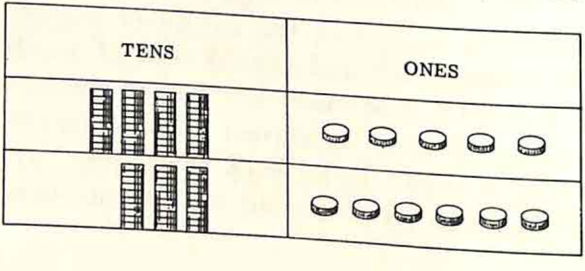
Cardinal Idea of Number. One of the first number concepts developed with young children is "how many?" This use of number has a special meaning, the cardinal idea of number. How do we teach a child what *three* means? First, by sets of objects, then by pictures of objects, and finally by the use of the abstract symbol 3.

By the use of one-to-one correspondence for sets of objects (a matching or pairing between two sets) the idea of equivalence of sets is learned. Three girls could be matched with three boys. Then other sets of objects equivalent to this set could be used to show that the nature of objects in each set may be disregarded, and the concept of number (how many?) remains. Ideas of *many* and *few* or *greater than* and *less than* should likewise be learned. The evaluation of pupil understanding is important for each new aspect. By observation the teacher can see if a child can use one-to-one correspondence to identify equivalent groups. The child can be asked to point to a set with many objects and then to a set with few objects. Another technique is to ask the child to make a circle or curved enclosure and *show* many marbles or marks. Then have him *show* less than his first picture in a second drawing. Finally, the child can be asked to tell how many balls are on the table, followed by writing the numeral *3* to avoid reversals and to insure correct formation of the numeral.

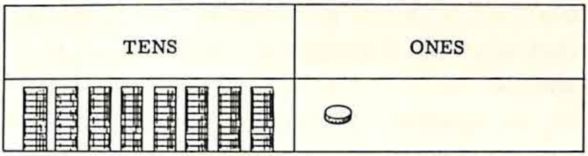
Thus, the evaluation involved in guiding the learning of *3* as a cardinal number included the understanding of basic vocabulary, application of "threeness" to many sets of objects, saying the number, making representations of the number, and writing the number (providing the teacher with an opportunity to observe coordination and dexterity).

Carrying to Ten's Position in Addition. This important concept could be introduced with a practical problem: How much money did we collect for the Red Cross during the last two days? One child, seeing the money in two boxes on the table, replies that he knows how much was collected one day: "I can see three dimes and six pennies and that's 36¢." The teacher accepts this reply and asks how much is in the second box. The children count the money in the other box and discover four dimes and one nickel—45¢. "Now what are we going to find?" asks the teacher. The children reply, "How much money is in both boxes?" By counting, the children discover that there is 81¢.

Moving from the concrete to semiconcrete, the teacher can have a child *show* 45¢ on a place-value chart (line one):



and 36¢ more on the same chart (line two). Still another child can complete the addition by making one bundle of ten from the ones column and carrying the one ten to the tens column. Thus 4 tens, 3 tens, and 1 ten carried makes 8 tens; only one remains in the ones column:

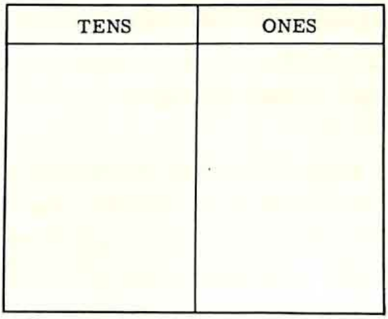


8 tens + 1 one = 81

After several problems are worked with the use of the pocket chart, the children are probably able to use individual charts and paper slips to work examples.

Examples:

$$\begin{array}{rrrr} 35 & 46 & 39 & 33 \\ +\ 7 & +\ 7 & +22 & +27 \\ \hline \end{array}$$



Pupil place-value chart made of 8″ × 11″ chipboard

The teacher can *observe* pupils' work with the pocket charts. If a child seems hesitant, the teacher can ask the child to *say* each step of the problem to determine how the child is thinking and working.

As a final step, probably on another day, the teacher can place these examples on the chalkboard:

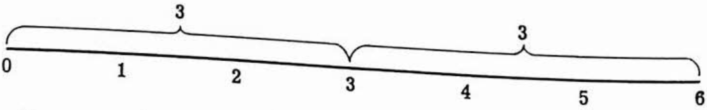
$$\begin{array}{rrrr} 37 & 58 & 67 & 35 \\ +44 & +32 & +14 & +47 \\ \hline \end{array}$$

This exercise should enable the teacher to obtain some measure of pupil understanding of the new process. If the work has been carefully presented in the first two steps, children will be able to carry the ten without using a crutch, $\begin{array}{r}37\\+44\\\hline 1\end{array}$, placing the one ten over the 3 to obtain an answer of 81. Children who cannot carry the one ten (by thinking) may not be ready for the abstract stage, or they may need to use the individual place-value chart a little longer.

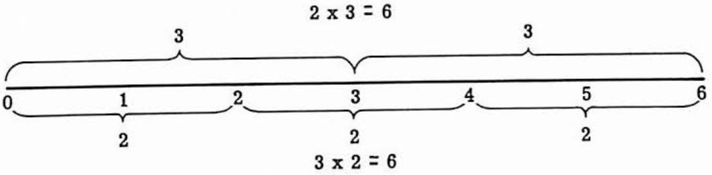
Evaluative techniques used in this sample lesson included the large place-value chart, money, and individual place-value charts. The aids were used to help children structure the problem and their thinking. Opportunity was provided for the teacher to observe pupils and to determine whether they understood the work at each stage of learning process.

Evaluation procedures for intermediate and upper grades

Identifying the Meaning of a Factor in Multiplication. Children must be taught the meaning of a multiplication fact such as $2 \times 3 = 6$. This is ordinarily taught in third grade. To test a child's understanding of what this fact means, the teacher can use a simple number line and have a child illustrate the fact:

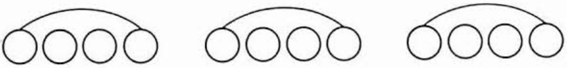


The diagram shows clearly the concept of multiplication as a process of cumulative addition with equal addends. Likewise, the commutative law of multiplication can be demonstrated with the number line thus:



Division Concepts. To test a child's understanding of measurement and *partition* in division, use markers or draw circles.

Ask the child to illustrate measurement using the example $12 \div 4 = 3$:



How many 4's are there in 12? There are three 4's in 12.

To illustrate partition using the same example: Divide 12 into 4 equal parts. What is the size of each part? The size of each of the equal parts is 3.

To find out how a child *thinks* in connection with the division process use this example:

$$\begin{array}{r} 6 \\ 43\overline{)2784} \\ \underline{258} \\ 20 \end{array}$$

Then ask the child several questions:

1. Why is the 6 placed over the 8 in the quotient?
2. Why do we work from left to right in division?
3. What is the value of 20 shown after subtracting 258 from 278?
4. Why do we bring down the 4 for the next step?

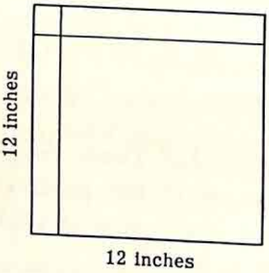
If division is taught as repeated subtraction, for example,

$$\begin{array}{r|l} 43\overline{)2784} & 60 \\ \underline{2580} & \\ 204 & \end{array}$$

similar questions may be asked to discover the child's mental processes as they are used in division.

Measurement Concepts—Area and Volume. It is advisable for the teacher to have cutouts of square inches, square feet, and square yards. This can be done with linoleum remnants. The child will then have an understanding of relative sizes of the units used in square measure. The real understanding of the meaning of area will be achieved as the pupil covers surfaces with these cutouts.

One sample test item to measure pupil understanding of area is: How would you draw a diagram to show how many square inches there are in one square foot? The child should have a diagram similar to this:



$$12 \text{ inches} \times 12 \text{ inches} = 144 \text{ inches}$$

Or he may actually draw out all lines to count 144 squares.

The volume of a box, or rectangular parallelepiped, can be measured directly. A box that is 2 inches by 3 inches by 4 inches can be separated into 3 horizontal layers. Each layer contains $2 \times 4 = 8$ cubic inches, by actual count. The total number of cubic inches is, therefore:

$$3 \times 2 \times 4 = 24.$$

The volume of the box is 24 cubic inches. As in the discussion on area, the teacher should have a box or block of wood representing the $2'' \times 3'' \times 4''$ so that children could actually count the cubic inches.

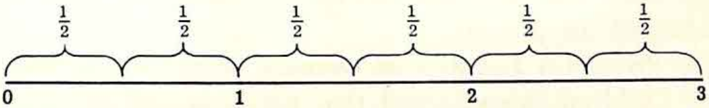
Simple but useful questions to measure pupil understanding of surfaces of the box just described are: how many surfaces does the box have? or, how many surfaces can you see if the box is placed on the desk?

Multiplication of Fractions. Diagrams of circles, rectangles, and line segments should be used to present new processes in working with fractions. For introductory work a circle may be used because a fractional part of a circle is easy to distinguish. A fractional part of a rectangle, removed from the original whole, may be mistaken for another whole rectangle.

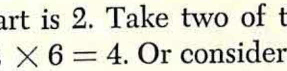
To discover whether a pupil understands this problem, $6 \times \frac{1}{2}$, the pupil should be able to tell how many. Thus:

$$6 \times \tfrac{1}{2} \text{ means } \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 3.$$

In this example the multiplication is treated as successive addition. Or the pupil may use a number line to show his answer.

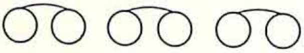


In working the example, $\frac{2}{3} \times 6$, the multiplier is a fraction. It is not an integer, so it cannot tell how many. This creates a difficult problem. How can we tell whether a pupil understands this process? If the process has been taught carefully and with understanding, the child should be able to reason thus: the $\frac{2}{3}$'s means that something has been broken into three equal parts and two of the equal parts have been taken. Applying this meaning to 6, divide 6 into three equal parts and then take two of them. If 6 is divided into three equal parts, the size of each part is 2. Take two of these equal parts. The result is 4. Then $\frac{2}{3} \times 6 = 4$. Or consider a diagram:

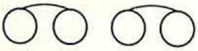


This illustrates 6. Now divide 6 into three equal parts:

Take two of these equal parts:



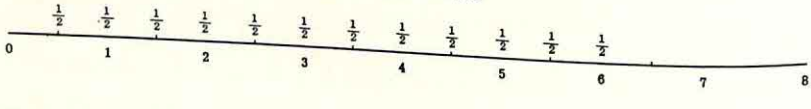
The result is 4. Thus, ⅔ × 6 = 4.



Division with Fractions. Division involving fractions is the most difficult process for teachers to present in a meaningful way. There is considerable evidence to show that many teachers do not fully understand the process themselves. For this reason they teach the process mechanically, using the time-worn rule "invert the divisor and multiply." Several suggestions for the teaching of division involving fractions will be given. Each step involves careful evaluation of pupil understanding of the process.

Use Principles Learned in Division With Whole Numerals First. Children should recall that partition and measurement concepts were used in division with whole numerals. So in the example, 6 ÷ ½, the work can be interpreted in two ways: 6 is divided into two equal parts; what is the size of each part? and how many twos are there in 6? In the example, 6 ÷ ½, the divisor is ½, which is a fraction. In this case, the language of *measurement* can be used. The language of partition cannot be used. Thus, 6 ÷ ½ may be explained on this basis: how many ½'s are there in 6? This can be illustrated with the number line

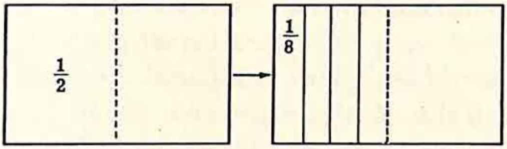
6 ÷ ½ = 12



By actual count there are twelve ½'s in 6, so 6 ÷ ½ = 12.

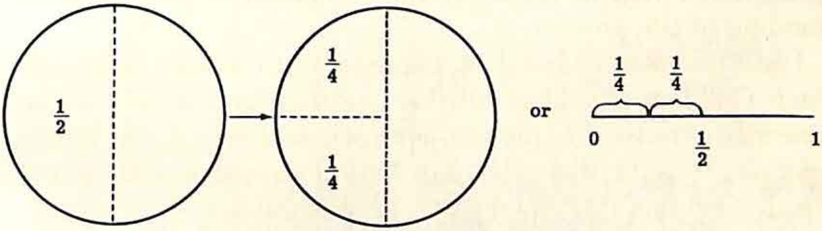
Consider the example ½ ÷ 4. The divisor is 4, which is an integer. In this case the language of *partition* can be used.

Thus, ½ ÷ 4 means: ½ is divided into four equal parts. *What is the size of each of the equal parts?*



½ ÷ 4 = ⅛

Now consider the example ½ ÷ ¼. The divisor is a fraction, ¼. The language of *measurement* must be used. How many ¼'s are there in ½? By this time children should be able to think the answer. To test their understanding, ask a pupil to make a drawing showing the correct answer. He may use a circle, rectangle, or number line.



There are two ¼'s in ½.

Other methods to work problems involving division with fractions should be used before introducing the inversion process. The author has found three methods which enable children to understand division with fractions.

1. Repeated subtractions. This is a simple process and is most useful in problems requiring the child to find a solution by thinking, "How many times can I subtract the divisor?" This, of course, is the idea of measurement. In the example 3 ÷ ½, the child can think: how many times can ½ be subtracted from 3, successively? In this case: 3 — ½ = 2½ —

$\frac{1}{2} = 2 - \frac{1}{2}$ . . . and so on. By actual count $\frac{1}{2}$ can be subtracted 6 times from 3, successively, so $3 \div \frac{1}{2} = 6$.

2. Inverse process. Children have learned that division is the inverse of multiplication. For example, $2\frac{1}{2} \div \frac{1}{2}$ can be rationalized in this way: what must I multiply $\frac{1}{2}$ by to get $2\frac{1}{2}$? The answer would be 5. This could easily be checked by thinking $\frac{1}{2} \times 5$ is $2\frac{1}{2}$. With this process, many children can be encouraged to use a variety of methods in solving mathematical problems. Flexibility in one's approach is highly desirable.

3. Common denominator. Another way to explain division of fractions is to change both fractions to a common denominator. Then:

$$\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \div \frac{2}{4}.$$

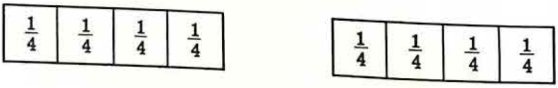
Now $\frac{3}{4} \div \frac{2}{4}$ is like 3 dollars divided by 2 dollars; it is 3 fourths $\div$ 2 fourths. The result is the quotient of the numerators

$$\begin{array}{r} 1\frac{1}{2} \\ 2\overline{)\,3} \\ \underline{2} \\ 1 \end{array}$$

or $3 \div 2 = 1\frac{1}{2}$.

The use of the familiar rule, invert the divisor and multiply, can be introduced after children understand division with fractions and have had experience using the three methods just described. This rule (invert and multiply) reduces division to multiplication.

To give meaning to this rule, consider the following approach: the example $2 \div \frac{1}{4}$ means—how many times is $\frac{1}{4}$ contained in 2? This can be illustrated:



$$2 \div \frac{1}{4} = 8$$

Each unit contains 4 fourths, so 2 units contain

$$2 \times 4 \text{ fourths} = 8 \text{ fourths}.$$

So: $2 \div \frac{1}{4} = 2 \times \frac{4}{1} = \frac{2}{1} \times \frac{4}{1} = 8$.

Now consider $2 \div \frac{2}{4}$. The divisor $\frac{2}{4}$ is twice as large as $\frac{1}{4}$, so $\frac{2}{4}$ will contain 2 only $\frac{1}{2}$ as many times as $\frac{1}{4}$ is contained in 2. Thus: $2 \div \frac{2}{4} = \frac{2}{1} \times \frac{4}{2} = \frac{8}{2} = 4$. To find how many times $\frac{1}{4}$ is contained in 2, simply multiply 2×4 or $2 \times \frac{4}{1}$. To find how many times $\frac{2}{4}$ is contained in 2, simply multiply by 4 and then divide by 2, or

$$2 \times \frac{4}{2} = \frac{8}{2} = 4.$$

By converting to improper fractions, all six cases found in the division process can fit the case: fraction ÷ fraction.

$$2 \div \frac{1}{2} = \frac{2}{1} \div \frac{1}{2} \qquad 4 \div 1\frac{1}{2} = \frac{4}{1} \div \frac{3}{2}$$

$$\frac{1}{2} \div 2 = \frac{1}{2} \div \frac{2}{1} \qquad 1\frac{1}{2} \div 4 = \frac{3}{2} \div \frac{4}{1}$$

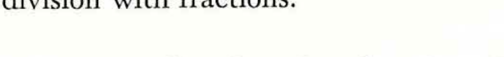
$$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \div \frac{1}{4} \qquad 4\frac{1}{2} \div 2\frac{1}{2} = \frac{9}{2} \div \frac{5}{2}$$

After these conversions have been made, the rule can be given: invert the divisor and multiply. Thus,

$$2 \div \frac{1}{2} = \frac{2}{1} \div \frac{1}{2} = \frac{2}{1} \times \frac{2}{1} = 4.$$

It should be apparent that the presentation of these division processes has been based upon a meaningful approach. Evaluation of pupil understanding should be an integral part of this process. There are, however, several simple techniques which can be used to collect information about pupils' understanding and application of division with fractions.

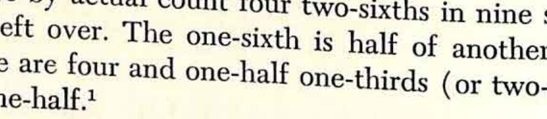
1. Which one shows $2 \div \frac{1}{3}$?



2. Show with circles, rectangle, or number line $2 \div \frac{1}{8}$.
3. Show with a rectangle or number line and tell what the answer (involves a remainder) means in $1\frac{1}{2} \div \frac{1}{3}$.

4. Make a word problem involving a whole numeral divided by a fraction.
5. Write a word problem showing a fraction divided by a whole numeral.

Exercises of this type provide ample opportunity for the teacher to determine whether children understand and can apply basic mathematical principles. In example 3 above, most prospective teachers are not able to work this problem and rationalize the process. When experiences are provided for them involving the use of the common denominator or successive subtraction methods, they grasp the meaning of this particular type of problem (29). When working the example $1\frac{1}{2} \div \frac{1}{3}$, use the measurement concept of division and ask, "How many one-thirds are in 1½?" Since the units are not the same, change 1½ to sixths and the ⅓ to sixths (using common denominator). So, $1\frac{1}{2} = \frac{9}{6}$ and $\frac{1}{3} = \frac{2}{6}$. Thus, we get $\frac{9}{6} \div \frac{2}{6}$. This can be rationalized with the use of a number line or rectangle:



$$\frac{9}{6} \div \frac{2}{6} = 4\frac{1}{2}$$

There are by actual count four two-sixths in nine sixths and one-sixth left over. The one-sixth is half of another division. Thus, there are four and one-half one-thirds (or two-sixths) in one and one-half.[1]

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[1] For detailed information on the use of line drawings, number lines, and lesson plans see: W. H. Dutton and L. S. Adams, *Arithmetic for Teachers.* Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1961.

EVALUATION PROCEDURES USED BY TEXTBOOK PUBLISHING COMPANIES

The revolution in mathematics has stimulated textbook publishers to revise their arithmetic series and to include suggestions for maintenance tests, diagnostic tests, and general evaluation procedures. Examples from several companies follow. The *Seeing Through Arithmetic Tests* have been designed for use with the *Basic Mathematics* program of Scott, Foresman & Company. Tests are designed for use in connection with Books 3 through 6. The tests have six parts including: problems involving selecting an answer, computation, selecting equations, solving equations, information, and concepts. The tests have been administered to a large number of pupils as part of a nation-wide experimental testing program. No fixed national norms are provided for the tests. Each school system using the tests are encouraged to develop their own norms. The publishers, however, do supply the users with results of national achievement testing so that general comparisons can be made each year. A cumulative individual record form is available to enable users to accumulate test results over a four-year period.

In the *Learning to Use Arithmetic* program of D. C. Heath & Company, inventory tests, diagnostic tests, maintenance tests, reviews, mastery tests, and problem-solving tests are placed at strategic points throughout the tests. The authors make the point that "to test is not necessarily to evaluate." It is the purpose for which the test is used and the interpretation of the test results in light of this purpose which constitutes an evaluation. The importance of continuous evaluation, involving both teacher and pupil, is stressed.

Another practical approach to evaluation has been developed by the Silver Burdett Company. In the *Modern Mathematics* series for Junior High School, attainment tests have been developed for each main chapter of the series. The tests are designed to help the teacher measure the extent to which students have understood the broad mathematical ideas of each chapter and mastered the computational skills involved. Each test has two parts: Part A tests understanding; Part B, computational skill.

CONCLUSIONS ON EVALUATION OF PUPIL UNDERSTANDING OF ARITHMETIC

The advantages of teacher-made tests and the use of informal evaluation procedures have been stressed in this chapter. The writer has given suggestions for the preparation of teacher-made tests and has included sample comprehension tests in the appendix for grades three through six.

The arithmetic comprehension tests found in the appendix have been used in four different school districts to show the need for this type of evaluative work and to present several methods for the interpretation and use of test data. The results of these tests pointed out the inescapable fact that mathematical understandings are not being adequately taught or evaluated.

In the last part of this chapter, suggestions for informal evaluation of pupil understanding of arithmetical concepts were given. Sample problems and teaching procedures were presented for evaluating pupil understanding as each new concept was introduced.

NOTES AND PROBLEMS

1. Discovering and removing pupil weaknesses in elementary school mathematics has not been a major goal of many teachers. After teaching a new algorithm to a group of school children or to children in your neighborhood, take time to measure their understanding of the new process. Use several methods—verbal

questions, test items from the comprehension tests, drawings, and child-prepared word problems using the new learnings. Make a list of pupil learning difficulties and misconceptions for the new algorithm taught.

2. Study one of the tables in this chapter used to show pupil response on an arithmetic comprehension test. Make an analysis of the arithmetical concepts you believe should be retaught and the remedial work needed.
3. Read reference (83) "Studying Arithmetic" and then list the ways teaching aids should be used with elementary school children to provide for individual differences in learning new arithmetical concepts.
4. Construct a set of line drawings to illustrate the main types of division of fractions.
5. What steps would you take to help children identify their own difficulties in working arithmetic problems? How could you help them to distinguish between mere computation and thorough understanding of a new process?

V

A Point of View on Evaluation

The process of meshing the old with the new has always been a challenging problem to the scientist and to the educator. If the discoveries and knowledge of the past are disregarded, there is a good chance that much time and effort may be lost in rediscovery or repeating work of merit. Merely rehashing old material in a new form, however, can be equally wasteful and may even be detrimental to creative thinking. With these considerations in mind, the writer has reviewed the historical background of the teaching of arithmetic and the efforts which have been made to change the content to meet the needs of pupils as well as the society of which they are a part. A good look has been taken at research studies and educational writings of authorities in the field of mathematics to discover data to support the recommendations the writer has made throughout the book.

PROBLEM AREAS

While the reader should have obtained important information pertaining to the development of mathematics in the elementary school up to this point, no position has been taken by the writer pertaining to the problem areas identified in Chapter I. Admittedly, there are other problems confronting educators, mathematicians, and psychologists interested in advancing the new mathematics. But the writer feels that the problems identified in this book are crucial and require immediate attention if sound progress is to be made in the teaching and learning of mathematics in the elementary school. One central theme should be easily identified in this book—teaching so that pupils understand each new mathematical concept and providing for inclusion of careful evaluation as an integral part of every teaching and learning experience connected with the new mathematics.

The new mathematics and the purposes of elementary education

The public elementary school was established to provide a general education for America's children. The customs, traditions, values, language, history, loyalties, and practices necessary for democratic living must be learned anew by each generation of children.

Changes which have taken place in the world have been caused, to a large extent, by the application of science and technology to economic relationships, conservation of human and natural resources, group and intergroup living, and international relations. Mathematics, one of the sciences, will continue to be one of the most important aspects of the technological revolution now going on in our nation as well as throughout the world. Demands for the new mathematics needed for effective living in modern America must be reflected in the educational objectives of the elementary school. To achieve these important goals, a specific kind of democratic education is required.

The curriculum design adopted by the school system controls the crucial decisions involved in selecting objectives, determining learning activities for children, and evaluating the total instructional program. There have been at least three basic approaches to curriculum organization for the elementary school: (1) subject-centered, (2) society-centered, (3) child-centered. The writer cannot be too emphatic in pointing out the necessity for achieving a good balance between these three approaches. Subject matter, concern for our democratic society, and the optimum development of each individual must be interwoven into a strong educational program.

The danger is great that the subject-matter approach to curriculum organization will be overemphasized in attempts to provide for the new mathematics at the elementary school level. In fact, considerable evidence is available to show that emphasis upon a strictly "mathematical approach," introduction of new forms of mathematics in many grades, and continued stress upon achievement measured by standardized tests have already begun to jeopardize some of the new programs. The necessity for including new mathematical concepts and programs is admittedly very important. The question is, how? Certainly not at the expense of other purposes of the elementary school! One has only to look at the crowded elementary school curriculum to realize that more emphasis upon subject matter will not further the purposes of general education. "Queen of the sciences, indeed!" Mathematics is well established in the elementary school program while science has been neglected. In fact, no other basic school subject has as insecure a standing as the elementary science program.

The research findings reported in this book have shown that learning mathematical concepts proceeds best when the pupil understands each new aspect. He must be able to examine the parts, reorganize or reassemble these parts into a meaningful whole or see their interrelationships, and from these experiences derive appropriate principles or generalizations. Along with understanding, the child must be provided with opportunities to use and apply new mathematical concepts, thus gaining some idea of their importance and developing skill in their application.

If a child at any grade level cannot understand a new concept and cannot apply newly learned concepts to the solution of problems, either the learning activity has been inadequately directed or the new concepts should be deferred until the child has gained more experience and maturity. These two factors, then, determine what children can learn of the new mathematics.

Placing appropriate emphasis upon subject matter, needs of children, and needs of our democratic society must be achieved through careful statement of educational objectives and adequate evaluation of the teaching-learning process so that these purposes are advanced.

Teacher preparation and the new mathematics

The research data presented in Chapters II and III point out the inadequacies of many elementary school teachers in their ability to understand basic mathematical and arithmetical concepts. This is not surprising since they were taught by a mechanically oriented approach in elementary school and received (in general) the same type of instruction in methods courses at the college or university levels. Future improvements in the education of prospective teachers of mathematics must be based upon an adequate analysis of student misconceptions and systematic efforts to assist each student to build new concepts based upon his present knowledge. The few available research studies indicate that elementary school teachers should examine their own understanding of the basic meanings in arithmetic.

Recommendations made by mathematical organizations suggest at least two years of college preparatory mathematics, consisting of a year of algebra and a year of geometry or the same material in integrated courses and a two-course sequence devoted to the structure of the real number system and its subsystems. These recommendations should be carefully examined in light of the evidence presented in this book. Individual differences among prospective teachers are just as great as the differences among the children they are preparing to teach. Some high school graduates, having had sound courses in algebra, geometry, and the structure of the real number system, will not need additional courses in these areas at the university level. They may need an advanced course in mathematics emphasizing new mathematical systems and providing them with the opportunity to learn appropriate teaching procedures at their own pace—a program entirely different from one planned for prospective teachers with less mathematical background and ability. The basic assumption underlying a twelve-semester unit requirement in mathematics for elementary school teachers is that this will insure qualified teachers. Nothing could be farther from the truth.

Emphasis must be placed upon insuring that prospective teachers understand the basic mathematical concepts they are expected to teach children rather than upon requiring additional courses or units, it seems to this writer. In achieving this end, drastic revisions will have to be made in the teaching at college and university levels. Teachers at this level will have to be prepared just as carefully as teachers needed in the elementary schools. Given well-prepared college or university teachers who will present mathematics courses for prospective teachers in a meaningful way, we may expect some students to complete their preparation largely on their own initiative and in one course or two courses at the most. Other students may require several courses and prolonged instruction (some of which need not be for college credit) to prepare them to teach elementary school mathematics in a meaningful way.

But what about the vast number of elementary school teachers who have completed their courses in mathematics and who hold a credential to teach in the public schools? Obviously, in-service education provided on a cooperative basis by teacher education institutions and local school districts must be planned. But before spending time on this important program for experienced teachers, another much more fundamental step must be taken. *School systems will have to place as much emphasis upon the measurement of pupil understanding of mathematical concepts as they are now placing upon achievement of basic facts and computational skills.* Tests measuring pupil understanding (similar to those provided in Chapter IV) must be required along with standardized achievement tests. Then, and only then, will many classroom teachers begin to examine their own understanding of mathematical concepts and teach so that children will understand the concepts being presented.

When administrators of public schools accept this recommendation, we may expect several innovations to take place in rapid succession. Those schools with adequate research staff will prepare valid, reliable instruments which will be used at each grade level to measure pupil understanding of basic mathematical concepts. Companies producing standardized achievement tests will eagerly begin preparing tests measuring pupil understanding. The lay public supporting education can be expected to move in the direction of supporting the evaluation process rather than being satisfied with the results of survey-type achievement tests. Finally, the classroom teachers will begin to prepare their own informal tests to measure pupil understanding of the mathematical concepts being taught. They will also use many techniques and instructional aids to test for pupil understanding as integral parts of the teaching-learning process.

Relationship of new mathematics to other school subjects

The historical background for the teaching of arithmetic was presented in Chapter I. General conclusions can be made from these historical data. There has been little disagreement concerning the extent to which mathematics permeates all aspects of the elementary school program. Concepts of time, space, quantity, and relationships are dealt with almost every day as children pursue their study of social studies, science, or even art and music. The usefulness of the mathematical concepts probably depends, to a large extent, upon the intelligence of the pupil, the background he has had with applying mathematics to his studies, and the careful guidance and concern of the classroom teacher.

Considerable disagreement has arisen over the extent to which mathematics should be integrated with other elementary school subjects. Three points of view were identified in the introductory chapter: (1) the complete lack of integration —mathematics as a separate discipline; (2) total integration through fusion of existing subjects; and (3) the need for synthesis—the mathematical and the social aspects dealt with in a balanced way.

Emphasis upon subject matter

One would think that the complete lack of integration, representative of the thinking of most mathematicians in the 1930's, had been dispelled. However, the writer has a real concern over the undue emphasis placed upon mathematical subject matter in the new programs. In their efforts to purge the school program of the heavy emphasis upon social application of mathematics, some mathematicians and proponents of the new mathematics are advocating too much mathematical sophistication. Along with this mathematical orientation has come the belief that most new mathematical programs will benefit all children. The disregard for individual differences and the failure to prepare at least three levels of mathematical materials for children is appalling to this writer. What is needed is purposeful teaching, by *teachers who understand mathe-*

matics and children, followed by careful evaluation to measure the accomplishments and depth of insights of their pupils.

Achieving a balance between social applications and mathematics

Achieving a balance between mathematics, per se, and the social applications of mathematics has important implications for the program of the elementary school. Two phases should be considered in this synthesized approach: (1) the contribution mathematics can make to learning in other subjects; and (2) the provision for a systematic approach in mathematical instruction which also uses some social application to motivate instruction and to aid in concept development. Neither phase is achieved without careful teacher guidance and appropriate curriculum planning.

Integration of mathematics with other subjects

If mathematical concepts are to be useful in learning activities in other school subjects, this transfer of mathematical knowledge must be specifically planned for in teaching. The inclusion of mathematical knowledge in an incidental way, haphazard or unplanned, probably injures mathematics and the subject with which it is integrated. There is entirely too much chance of misconceptions being learned and too much time consumed in the process. If there is a significant place for mathematics, such as the understanding of time belts (both mathematical and scientific) in studying transportation in social studies, the concepts must be taught with understanding and mastered before effective learning will take place. Thus, the contributions of mathematics are crucial to the social studies program and must be dealt with carefully. Indispensable mathematical concepts needed to enrich the work and to bring about desirable learning in any elementary school subject require planning.

Contributions of social situations to mathematical learnings

The use of social situations, content from other fields, or instructional aids involving social implications in the learning of mathematical concepts has often been poorly handled by teachers. This has been so because of misconceptions held by teachers and because of the dearth of research data on the use of social content in the development of mathematical concepts. There are, however, important research findings which need reporting and implementation in mathematical instruction in elementary schools.

1. The use of a social situation (the amount of concrete needed for a new school tennis court) will probably motivate junior high school pupils to study the mathematical concepts of volume or area. A reason for study and the will to study are important and desirable to learning. But the individual child must learn the mathematical concept which is part of the logical structure of our mathematical program. It is possible that too much time may be spent on the social situation and not enough time spent on the mathematical. Care must be taken in the selection of the social situation so that the social and the mathematical concept to be learned fit closely.

2. Similarly, the use of instructional aids involves problems of appropriateness of the aid and the structuring of the child's thinking as he uses the aids. A place-value chart or board is indispensable, according to this writer, in the teaching of regrouping in subtraction at the third-grade level. There is some evidence to indicate that with the use of a place-value aid children will not need to use a crutch (marking out numerals) in beginning subtraction work. Children must, however, be able to handle this algorithm mentally. Once having learned

the new step, the aid does not need to be repeatedly used for some or even many of the children. As the writer has pointed out before in Chapter II, both the verbal language and the instructional aid used by the teacher must be appropriate for the concept being taught and for the thinking of the child doing the learning!

3. One way to test a child's understanding of a new mathematical concept is to ask him to make a social problem using the newly learned concept. Translating problem situations into their own language or into appropriate mathematical models is helpful for children and gives the teacher some indication of pupil understanding. Again, the mathematical thinking and the ability to do abstract thinking is the outcome sought—not the manipulation of a model or the use of a social phenomenon.

4. Rigidly structuring a new mathematical concept for the child is dangerous. This has been a common mistake made by educators and mathematicians in the past. A variety of methods for solving problems should be encouraged and taught. The teacher may gain some measure of pupil understanding and the thinking process used by asking a pupil to work the problem with another method, using an inverse process, or even combining methods to secure a reasonable answer.

Thus, as in the other suggestions, the social application, use of aids involving a social implication, or language used to convey mathematical thinking are supplementary to the actual learning of the mathematical structure involved. In order to learn how to handle mathematical concepts at increasingly higher levels of abstraction and sophistication, the learner must understand each new aspect thoroughly. The teacher's task is to be able to evaluate each aspect of the learning situation and to help the individual gain these understandings at a pace commensurate with the child's needs and ability. Yes, arithmetic has both a mathematical and a social aim. Children must see sense in the arithmetic they learn and must understand the structure of arithmetic as mathematics.

Learning theories and the new mathematics

The meaningful approach to teaching arithmetic received its original impetus from psychologists who were proving the advantages of learning with understanding instead of learning by isolated drill. The major development during the past quarter of a century has been the discovery of how this concept of meaningful learning applies to arithmetic teaching.

The leaders of the revolution in mathematics have been mathematicians, and the experimental programs they have sponsored have been in the direction of greater mathematical sophistication. The position held by many of these leaders has been that we can teach much more mathematics to pupils of any given age than we are now doing. As for learning theories developed and extended by leading psychologists, a seemingly new term, *discovery*, has been used to describe or even replace the meaningful approach. The contention of ardent supporters of the new mathematics has been that with the discovery method a pupil learns arithmetic even faster and retains it even longer than with the meaningful approach. With few exceptions, the new mathematics programs have been directed toward the above-average child and even toward homogeneous grouping. The provision for individual differences, as the writer has pointed out before, has not been a major goal of the supporters of the new mathematics. What, then, are some ways to bring psychologists and mathematicians closer together? What can be done to guide and accelerate progress in the new mathematics?

The main purpose of this book has been to supply data which will help in the solution of these problems. There is ample evidence available to point out the advantages of meaningful arithmetic. Much progress has been made in identifying the concepts, principles, relationships, and generalizations essental to learning arithmetic. Yet, as the writer has shown, a tremendous gap exists between what is known about learning theory and what is being applied in the teaching of arithmetic in elementary schools throughout the nation. This writer suggests a few steps which should be taken immediately if educators and mathematicians are going to guide the new mathematics along constructive lines:

1. All experimental programs now in progress should have well-balanced consultant teams including educational psychologists, mathematicians, and curriculum experts to provide guidance for the preparation of instructional materials and to work on the statement of objectives.
2. Clearly defined objectives should be prepared for each new aspect of the new mathematics. These objectives should be stated in terms of desired pupil behavior.
3. Careful attention must be given to the preparation and use of techniques which will provide for evaluation of the *total range* of objectives established for each program.
4. Instructional materials should be prepared for at least three learning levels: (1) gifted and advanced pupils, (2) the average child, and (3) the slow learner. Then adequate experimentation should be provided to see how pupils of varying abilities learn the mathematical concepts presented to them. Careful evaluation based upon specific objectives must be an integral part of this experimentation.
5. A variety of instructional materials and aids should be prepared and tried out. This should include programmed material, machine teaching, television, motion pictures, and tactile, visual aids. Again, careful evaluation based upon specific objectives for the use of these materials must be provided.
6. Local school districts and state departments of education should be encouraged to try out new mathematical programs by adapting them to their local situations. As in the first step, a consultant team (including a psychologist, mathematician, and curriculum expert) should be made available to

guide the work and to insure appropriate evaluation of the results.

7. A careful distinction should be made between programs which are designed to try out and develop instructional materials for the elementary schools and projects which are largely experimental. While there is much to be gained from experimental work, and there are many elementary mathematical concepts young children can learn, the general effect may be to encourage teachers and parents to expect children to master too much abstract mathematics.

The discussion so far should not be construed by the reader to mean that the writer is opposed to the work being accomplished by the mathematical revolution now under way. This is far from the truth. The writer is highly concerned about child growth and development, yet there are precious few of the leaders in this area working with the new mathematical program. The research evidence on learning theories, development of creative abilities, and problem-solving is respectable and growing yearly; yet, few psychologists of national reputation have been involved in the preparation of the new arithmetic materials. Evaluation techniques have been developed and improved over the last two decades, yet most experimental programs and regular mathematics programs are woefully weak or entirely neglectful of this important area. Just as important is that other areas of the elementary school curriculum are badly in need of revision (particularly elementary school science), and the patterns now being used to "hew" out new mathematical programs will be used to make similar revisions in other content areas. There is yet time to take a careful look at the procedures now being used and to provide the working organization for a much more successful approach to curriculum improvement than we have known to date. With more emphasis placed on a desirable design for curriculum improvement and with valid evaluation procedures carefully applied, the results of the mathematical revolution now under way may exceed even the highest expectations of the ardent mathematicians supporting the newer programs.

Evaluative instruments for the new mathematics

The necessity for using desirable evaluation procedures has already been discussed in connection with teacher preparation, sound learning theory, and appropriate curriculum improvement. Consideration will now be given to the evaluative process within the elementary school classroom.

The administrative considerations, discussed earlier in this chapter, would direct teachers' efforts toward evaluation of the comprehensive range of objectives. This would constitute an important first step. The success or failure of evaluative work, however, will be determined by what each teacher does about this important process. The suggestions which follow are, according to this writer, necessary for desirable teaching and the success of modern mathematical programs:

1. Teachers should begin working on evaluation by establishing clearly defined purposes for fundamental processes which they are expected to teach. Using one or more of the test items provided in this book should enable them to test their own understanding of the new concept as well as their pupils.
2. Evaluative techniques, such as asking a pupil to "say the problem" aloud as he works it, to demonstrate the meaning of a process with the use of an aid or model, or to make a word problem of his own, should be used as new concepts are being taught.
3. Some record should be kept of the progress children make toward understanding new arithmetical concepts. Because of individual differences in ability, background in mathematics, and environmental factors, pupils will progress at different rates. This being so, adequate pupil learning and effective teacher guidance will be dependent upon the continuity made possible by a record which will show the pupil and the teacher the progress being made.
4. Teachers should be searching for new ways to evaluate pupil understanding of the new mathematical concepts introduced into the elementary school curriculum. This should enable the teacher to keep

pace with modern developments by constantly testing his own understanding of the new concept as well as to motivate him to use new teaching procedures.

NOTES AND PROBLEMS

1. A study of the contributions of psychological research to the teaching of mathematics was made by Washington University in cooperation with the U.S. Office of Education. This project culminated in a conference on Psychological Problems and Research Methods in Mathematics Training, held at Washington University May 1-2, 1959. The papers given at the conference are contained in a monograph (34). Make a list of the problem areas identified by workers on this project. What recommendation do you have for carrying out research work in the problem areas identified?
2. Compare the author's list of problem areas with the list identified by the conference on psychological problems. What major likenesses and differences do you discover?
3. What steps do you think should be taken to bring about desirable evaluation procedures for modern mathematical programs?
4. What value do you see in modern research programs dealing with the teaching of elementary school mathematics?
5. Prepare a brief statement of your own point of view toward elementary school mathematics.

VI

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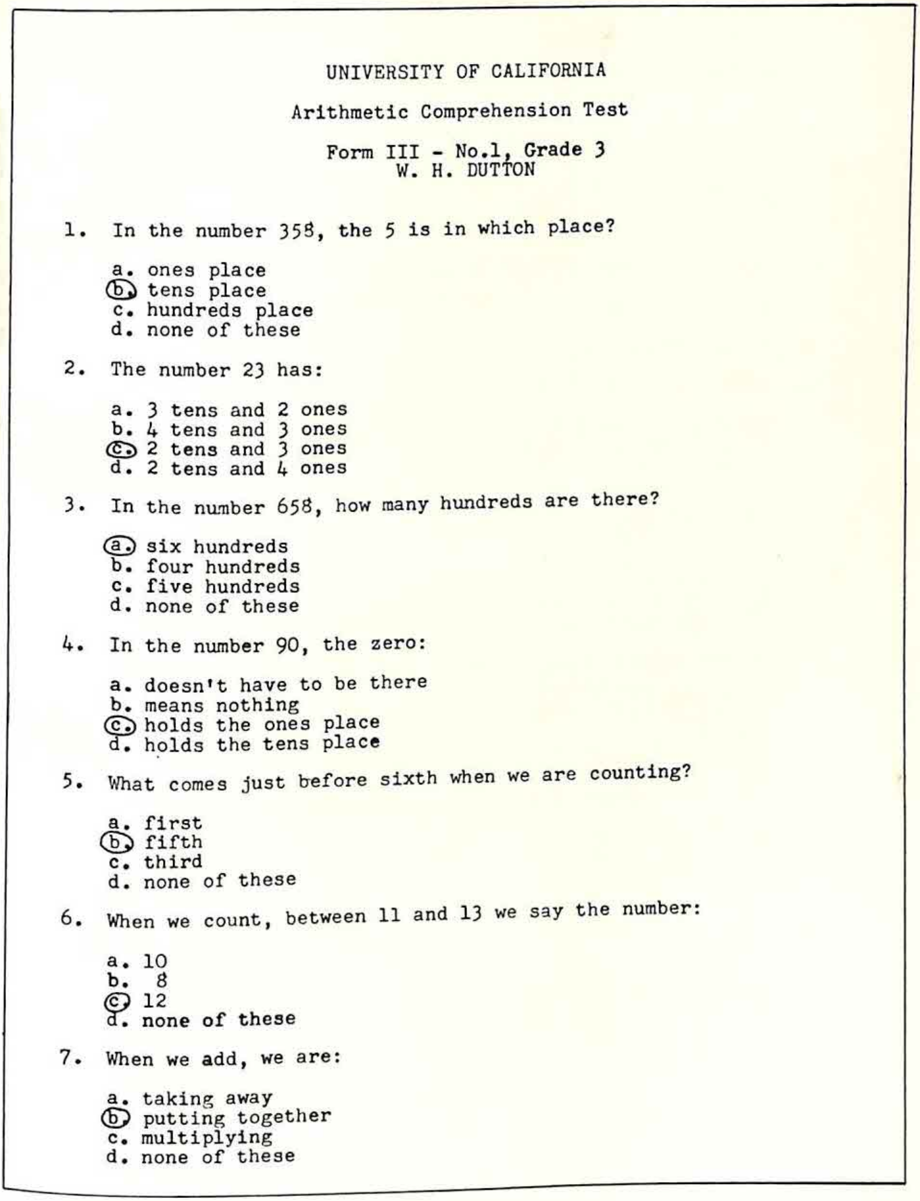
Appendix



UNIVERSITY OF CALIFORNIA

Arithmetic Comprehension Test

Form III - No.1, Grade 3
W. H. DUTTON

1. In the number 358, the 5 is in which place?

   a. ones place
   [✓ b.] tens place
   c. hundreds place
   d. none of these

2. The number 23 has:

   a. 3 tens and 2 ones
   b. 4 tens and 3 ones
   [✓ c.] 2 tens and 3 ones
   d. 2 tens and 4 ones

3. In the number 658, how many hundreds are there?

   [✓ a.] six hundreds
   b. four hundreds
   c. five hundreds
   d. none of these

4. In the number 90, the zero:

   a. doesn't have to be there
   b. means nothing
   [✓ c.] holds the ones place
   d. holds the tens place

5. What comes just before sixth when we are counting?

   a. first
   [✓ b.] fifth
   c. third
   d. none of these

6. When we count, between 11 and 13 we say the number:

   a. 10
   b. 8
   [✓ c.] 12
   d. none of these

7. When we add, we are:

   a. taking away
   [✓ b.] putting together
   c. multiplying
   d. none of these

8. This is an addition problem with:

$$\begin{array}{r}123\\+416\\\hline\end{array}$$

a. one-place numbers
b. two-place numbers
[✓ c. three-place numbers]
d. none of these

9. Three girls were playing ball. Two boys came to join them. How many children were there altogether? Would you:

a. subtract
[✓ b. add]
c. multiply

10. The clue to the problem above is:

a. playing ball
b. how many
[✓ c. altogether]

11. What kind of a problem is this?

$$\begin{array}{r}3\\4\\1\\2\\\hline\end{array}$$

[✓ a. addition problem]
b. subtraction problem
c. multiplication problem
d. none of these

12. This problem asks you to add:

$$\begin{array}{r}24\\+\ 2\\\hline\end{array}$$

a. 2 two-place numbers
b. 2 one-place numbers
[✓ c. a one-place number and a two-place number]
d. a one-place number and a three-place number

13. In the problem $\begin{array}{r}14\\+18\\\hline\end{array}$, what number is carried?

a. 1 one
b. 2 ones
[✓ c. 1 ten]
d. none of these

14. When we subtract, we are:

a. putting together
[✓ b. taking apart]
c. multiplying
d. none of these

15. $\begin{array}{r}87\\-15\\\hline\end{array}$ To subtract in this problem, we must:

a. make the 7 ones larger
b. make the 1 ten larger
c. make the 5 ones larger
[✓ d. none of these]

16. 82 To subtract in this problem, we must:
-17

[✓ a.] make the 2 ones larger
b. make the 7 ones larger
c. make the 1 ten larger
d. none of these

17. Seven children were playing together. Two children had to go home. How many children were left? Would you:

[✓ a.] subtract
b. add
c. multiply

18. The clue to the problem above is:

a. playing together
b. at the playground
[✓ c.] how many were left

19. 48 To subtract in this problem, we must:
-19

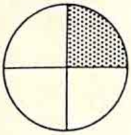
a. carry
[✓ b.] regroup the 4 tens
c. regroup the 1 ten
d. subtract without doing anything else first

20. Eight seals were playing on a rock. When two seals fell into the water, how many remained on the rock?

Would you:	The clue is:
a. add	d. seals were playing
b. multiply	[✓ e.] how many remained
[✓ c.] subtract	f. on the rock

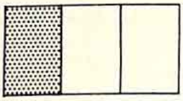
21. The colored part of the circle is what part of the circle?

a. $\frac{1}{2}$
[✓ b.] $\frac{1}{4}$
c. $\frac{1}{3}$
d. none of these



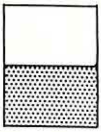
22. The colored part of this box is what part of the box?

a. $\frac{1}{2}$
b. $\frac{1}{4}$
[✓ c.] $\frac{1}{3}$
d. none of these



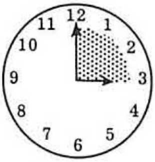
23. The colored part of this box is what part of the box?

a. $\frac{1}{4}$
b. $\frac{1}{3}$
[✓ c.] $\frac{1}{2}$
d. none of these



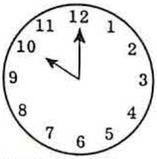
24. The hands on this clock form what part of the clock face circle (colored in)?

a. $\frac{1}{2}$
b. $\frac{1}{3}$
[✓ c.] $\frac{1}{4}$
d. none of these



25. The time on this clock is:

[✓ a.] 10 o'clock
b. 4 o'clock
c. 6 o'clock
d. none of these

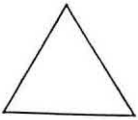


26. When the short hand of the clock is between the seven and the eight; and the long hand is on the six, what time is it?

a. seven o'clock
b. eight o'clock
[✓ c.] half past seven
d. half past eight

27. This is a picture of a:

a. square
[✓ b.] triangle
c. circle
d. none of these



28. A quart is:

[✓ a.] more than a pint
b. less than a pint
c. the same as a pint
d. none of these

29. Which is longest?

[✓ a.] mile
b. yard
c. inch
d. foot

30. IV in Roman numerals is the same as:

a. 3
b. 10
c. 9
[✓ d.] 4

31. XII in Roman numerals is the same as:

a. 14
b. 11
[✓ c.] 12
d. 9

32. November comes in which season of the year?

a. spring
[✓ b.] autumn
c. winter
d. summer

33. This sign means dollars:

[✓ a.] $
b. ¢
c. *
d. none of these

34. Three dimes and 4 cents are how many cents?

a. 45 cents
b. 64 cents
[✓ c.] 34 cents
d. none of these

Arithmetic Comprehension Test
Form 1B Grade 4

W. H. Dutton
University of California

- -

Instructions to pupils: This test will show how well you understand arithmetic. Answer all items. Usually there is one best answer. Place the letter representing the correct choice of answers on the answer sheet.

- -

PLEASE DO NOT MARK ON THIS TEST.

Sample exercise: The shortest time is:

a. day
[✓ b.] second
c. minute
d. year

- -

1. The most money is:

a. 75 cents
[✓ b.] 50 nickels
c. 10 dimes
d. 2 quarters

2. The answer to the problem $\begin{array}{r}42\\+10\\\hline 52\end{array}$ is called:

a. product
[✓ b.] sum
c. remainder
d. quotient

3. In the number 90, the zero:

a. doesn't have to be there
b. means nothing
[✓ c.] holds the one's place
d. none of these

4. The oldest kitten is:

a. 25 days old
b. 1 month old
c. 7 weeks old
[✓ d.] ½ year old

5. The most money is:

a. $1.40
[✓ b.] $4.10
c. $4.01
d. $0.41

6. When writing money, the two places to the right of the decimal are:

a. fractions
[✓ b.] cents
c. zeroes
d. dollars

7. Dozen means:

a. a trio
b. a pair
[✓ c.] twelve
d. to count

8. ½ of these 0 0 0 0 0 0 is:
0 0 0 0 0 0

a. 4
[✓ b.] 6
c. 3
d. 2

9. Seven million, eight hundred four thousand forty five is written:

a. 7,840,450
[✓ b.] 7,804,045
c. 7,840,045
d. 7,800,045

10. When writing money, the numbers to the left of the decimal are:

a. answers
b. cents
c. signs
[✓ d.] dollars

11. The longest time is:

a. month
[✓ b.] year
c. week
d. second

12. The most weight is:

[✓ a.] a ton
b. an ounce
c. a load
d. a pound

13. Perimeter means:

a. distance across any surface
b. surface of anything
[✓ c.] distance around the outside of any surface
d. the total distance of two surfaces

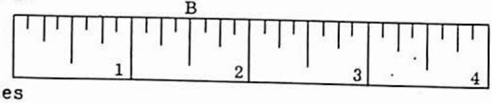
14. Temperature is measured in:

a. pounds
[✓ b.] degrees
c. hours
d. days

15. Eighths mean:

a. equal parts of any number
b. some equal parts of a whole
c. six and two
[✓ d.] eight equal parts

16. Look at the ruler. What part of an inch is shown by the mark under the B?

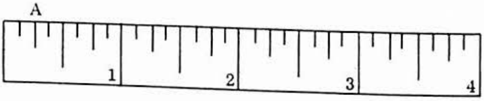


B is at:
a. 2 inches
[✓ b.] $\frac{1}{2}$ inch
c. 3 inches
d. $1\frac{1}{3}$ inches

17. The largest measure is:

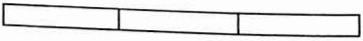
a. half gallon
[✓ b.] a gallon
c. a quart
d. a pint

18. Look at the ruler. What part of an inch is shown with the line under the A?



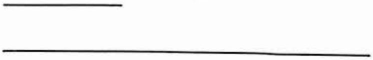
A is at:
a. 1 inch
b. $\frac{1}{2}$ inch
[✓ c.] $\frac{1}{4}$ inch
d. $\frac{3}{4}$ inch

19. This line is cut into:



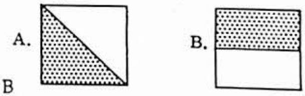
a. halves
b. fifths
c. fourths
[✓ d.] thirds

20. Look at the two lines below. What part is the top line of the bottom line?



a. $\frac{1}{2}$ of the bottom line
[✓ b.] $\frac{1}{3}$ of the bottom line
c. $\frac{1}{4}$ of the bottom line
d. $\frac{1}{8}$ of the bottom line

21. Look at the squares. The shaded part of square A is:



[✓ a.] equal to the shaded part of square B
b. more than the shaded part of square B
c. half of the shaded part of square B
d. Twice the shaded part of square B

22. Look at the Roman Numeral IX. The I before the X means:

a. X plus one
[✓ b.] X minus one
c. X times one
d. X divided by one

23. Look at the Roman Numeral VI. The V before the I means:

a. 5 times one
b. 5 divided by one
c. 5 minus one
[✓ d.] 5 plus one

24. Look at the Roman Numeral XX. The XX means:

[✓ a.] 10 + 10
b. 10 X 10
c. 10 - 10
d. 10 ÷ 10

25. In the problem $\begin{array}{r} 40 \\ \underline{--} \\ 14 \end{array}$ the missing number is found by:

a. dividing
b. adding
c. multiplying
[✓ d.] subtracting

26. The answer to the problem $\begin{array}{r} \$1.40 \\ \underline{-\ 1.38} \end{array}$ will be:

a. dollars
b. zeroes
[✓ c.] cents
d. dollars and cents

27. To check a multiplication problem:

a. add the product to the number multiplied
b. multiply the product by the multiplier
[✓ c.] divide the product by the multiplier
d. none of these

28. When the short hand of a clock is between the seven and the eight, and the long hand is on the six, what time is it?

a. seven o'clock
b. eight o'clock
[✓ c.] half past seven
d. half past eight

29. When multiplying the problem $\begin{array}{r}385\\ \text{X}\ \ \underline{\ \ 5}\end{array}$, we say:

a. 5 times 5 ones are 25 ones and write 25 in the one's place
b. 5 times 5 ones are 25 tens and we carry 2 hundreds
c. 5 times 5 ones are 25 hundreds and we carry 2 ones
[✓ d.] 5 times 5 ones are 25 ones and we carry 2 tens

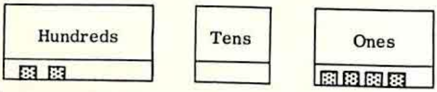
30. When dividing the problem $4/\overline{16}$, we think:

a. divide 1 by four
[✓ b.] 1 ten will not divide into 4 equal groups of ten
c. what number subtracted from 16 equals four
d. sixteen is to be multiplied by 4

31. This problem $7/\overline{21}$ means:

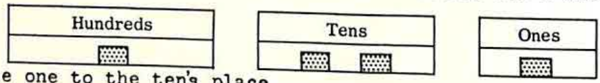
[✓ a.] 21 is to be divided into 7 equal groups
b. 7 is to be divided into 21 equal groups
c. 21 is to be divided into three equal groups
d. 21 times seven

32. Look at the place-value chart. What number does the chart show?



a. 214
b. 240
c. 24
[✓ d.] 204

33. Look at the place-value chart. To take 2 ones from the 1 one you must:



a. take the one to the ten's place
b. cannot do it in any way
[✓ c.] take a ten from ten's place and regroup
d. take a hundred from hundred's place

34. One is how many times as large as .001?

(a) 10 [✓ (b)] 1,000 (c) 100 (d) 10,000

35. In \$.10$\frac{1}{2}$, what does the $\frac{1}{2}$ mean?

[✓ (a)] half of one cent (b) five cents (c) eleven cents
(d) none of these

36. Two hollow tubes have exactly the same inner radius. Their volumes will be:

a. the same
[✓ b.] will vary directly as their lengths
c. will vary inversely as their lengths
d. equal to the radius plus the length
e. none of the above

37. Why is the decimal point placed after the 3?

$$\begin{array}{r} .52 \\ 1.42 \\ \underline{1.32} \\ 3.26 \\ \nwarrow \end{array}$$

a. keep the decimal point in a straight line
b. six places divided by 3 = 2 places
[✓ c.] hundredths added to hundredths point off hundredths
d. none of these

38. Why is the decimal point placed before the 6?

$$\begin{array}{r} 1.25 \\ \underline{\text{X} \quad .5} \\ .625 \\ \nearrow \quad \end{array}$$

[✓ a.] tenths times hundredths = thousandths
b. counting all decimal places you get three
c. you are multiplying by .50, so keep the decimal point in a straight line
d. none of these

39. A sixth grade class used 3 reams of paper per month. If there are 30 pupils in the class, what is the approximate average number of pieces of paper used by each pupil each month?

(a) 60 (b) 120 (c) 30 [✓ (d)] 50 (e) none of these

40. 2 X 1$\frac{1}{2}$ = (a) 1 (b) 2 [✓ (c)] 3 (d) 4

41. $\frac{5}{6} \div \frac{1}{8}$ would be: (a) $\frac{5}{48}$ [✓ (b)] 6$\frac{2}{3}$ (c) $\frac{6}{40}$ (d) none of these

42. Bill takes 200 pieces of candy from a box that holds 200 pieces. What per cent did he take?

(a) 400% (b) 200% [✓ (c)] 100% (d) 50%

43. Place the answers to the problems below on the ruler shown on the answer sheet.

a. at 2.5 in. put an X
b. at 1.75 in. put an O
c. at .75 in. plus .50 in. put a ✓

44. One-half times one-half (a) $\frac{1}{2}$ [✓ (b)] $\frac{1}{4}$ (c) $\frac{2}{2}$ (d) $\frac{2}{4}$

45. In working the problem below, which step needs to be taken to solve the problem?

$$\begin{array}{r} 3\text{ hr. }\ 40\text{ min. }\ 10\text{ sec.} \\ -\ \underline{1\text{ hr. }\ 50\text{ min. }\ 12\text{ sec.}} \end{array}$$

a. regroup seconds and minutes
b. add all elements and subtract
c. change all factors to hours and subtract
[✓ d.] regroup hours, minutes, seconds
e. none of these

46. In which of the following would area in measurement be used?

a. the shape of a ball
b. the height of a ceiling
c. the length of a football field
[✓ d.] the size of a floor
e. none of these

Arithmetic Comprehension Test
Form IB Grade 4 4-60

W. H. Dutton
University of California

Page 1
Sample ____

Please print
each answer

1. ____
2. ____
3. ____
4. ____
5. ____
6. ____

Page 2
7. ____
8. ____
9. ____
10. ____
11. ____
12. ____
13. ____
14. ____

Page 3
15. ____
16. ____
17. ____
18. ____
19. ____
20. ____

Page 4
21. ____
22. ____
23. ____
24. ____
25. ____
26. ____

Page 4 (Continued)
27. ____
28. ____

Page 5
29. ____
30. ____
31. ____
32. ____
33. ____

Page 6
34. ____
35. ____
36. ____
37. ____
38. ____
39. ____
40. ____

Page 7
41. ____
42. ____
43. ____
44. ____
45. ____
46. ____
47. ____
48. ____

Name ________
1. Boy ____ Girl ___
2. Age Yrs.___ Mo.___
3. School_ _ _ _ _
4. ___________
5. ____ ____
Reas. Fund

6. 1 2 3 4 5
low high

Arithmetic Comprehension Test
Form I A, Grade 5

W. H. Dutton
University of California

Instructions to pupils: This is an arithmetic test used to discover how well you understand arithmetic. Answer all items. work as fast as you can without making mistakes. Your teacher may pronounce words or read a test item for you.

Place your choice of answers on the answer sheet. Do not mark on the test booklet.

1. Round off 204,660 to the nearest thousand.

    a. 200,000
    b. 200,400
    c. 204,000
    [✓ d.] 205,000

2. Which number is larger than 3.046?

    a. 3.045
    b. 3.039
    [✓ c.] 3.100
    d. 3.014

3. Which number is less than 1.020?

    a. 1.035
    b. 1.100
    c. 1.039
    [✓ d.] 1.014

4. One-third of one is:

    a. $1\frac{1}{3}$
    [✓ b.] $\frac{1}{3}$
    c. $\frac{3}{1}$
    d. 1

5. Show $3\frac{2}{3}$ another way.

    a. $3\frac{5}{3}$
    b. $2\frac{2}{6}$
    c. $2\frac{4}{6}$
    [✓ d.] $2\frac{5}{3}$

6. How many eggs are there in $\frac{1}{4}$ dozen?

    a. 6
    b. 4
    [✓ c.] 3
    d. 5

7. Three boys received $1.80 for working on a lady's yard. How much was each boy's equal share?

a. $.90
[✓ b.] $.60
c. $.65
d. $.75

8. Mr. Jones spends $12.25 a month for gasoline. How much does he spend in one-half year?

a. $75.00
[✓ b.] $73.50
c. $60.50
d. $72.80

9. Two and one-half subtracted from four and one-half will equal:

a. two and one-half
b. one-half
c. six and one-half
[✓ d.] two

10. To reduce fractions to lowest terms we:

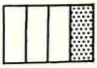
a. divide the numerator by the denominator
b. divide the denominator by the numerator
[✓ c.] divide the numerator and denominator by a common divisor
d. all of these

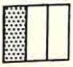
11. In the problem $2\frac{3}{8} + 4\frac{6}{8}$ we:

a. add the fractions first
b. add the whole numbers first
c. add the fractions but do not change them
[✓ d.] add the fractions and carry the whole to the one's column
e. do both a and d

12. Look at the boxes. Which one shows $\frac{1}{3}$ shaded?

a. box 1
[✓ b.] box 2
c. box 3
d. all of them


1.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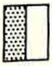

3.

13. Three-thirds plus four-fourths is:

a. seven-sevenths
b. $\frac{7}{4}$
[✓ c.] 2
d. 77

14. In the number 7,404, the 4 on the left is what relation in value to the 4 on the right ?

[✓ a.] 100 times larger
b. 10 times larger
c. 1,000 times larger
d. $\frac{1}{100}$ of it

15. In the number 86,469, the 6 on the right is what relation to the 6 on the left?

[✓ a.] $\frac{1}{100}$
b. 100 times larger
c. $\frac{1}{10}$
d. 10 times larger

16. Look at the sample problem. What does the 208 mean?

$$\begin{array}{r} 34\overline{)5482} \\ \underline{34} \\ 208 \end{array}$$

[✓ a.] Two thousand eighty
b. two thousand eight
c. two hundred eight
d. none of these

17. Look at the Roman Numeral IX. The I means:

a. X plus one
b. X times one
[✓ c.] X minus one
d. X divided by one

18. Look at the Roman Numeral VI. The V means:

[✓ a.] 5 plus one
b. 5 times one
c. 5 divided by one
d. 5 minus one

19. To multiply five by eight is to:

a. increase five by eight
b. add five eights
c. increase five by thirteen
[✓ d.] add eight fives

20. One dime and what coin equal 15 cents?

a. dime
b. quarter
[✓ c.] nickel
d. penny

21. The base of our Arabic number system is:

a. 1
b. 5
[✓ c.] 10
d. 100

22. Hal had 24 pencils. He broke 7. How many were not broken? What should you do to find the answer?

a. add
[✓ b.] subtract
c. multiply
d. divide

23. There were 8 rows of desks in a room with 5 desks in each row. How many children could sit in the room if each sat in one desk? How would you find the answer?

a. add
b. subtract
[✓ c.] multiply
d. divide

24. John had 40 pieces of candy and wanted to give each of his 5 friends an equal number. How many did each friend get? How would you find the answer?

a. add
b. subtract
c. multiply
[✓ d.] divide

25. In the multiplication example given below, the error is due to:

[✓ a.] carrying
b. adding
c. multiplying
d. place of zero

$$\begin{array}{r} 842 \\ \text{X } 54 \\ \hline 3,368 \\ 4,210 \\ \hline 7,578 \end{array}$$

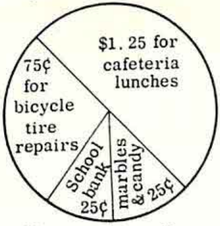
26. In the division example given below, the quotient will begin in what column?

a. tens
[✓ b.] hundreds
c. thousands
d. ten thousands

$$40\overline{)35780}$$

27. Jimmy had a job mowing lawns so that he had quite a bit of money every week. The diagram shows how he spent it. For which item does he use half of his money?

a. money saved to repair his bicycle tire
[✓ b.] money used for cafeteria lunches
c. money saved for the school bank
d. money spent for marbles and candy



28. Jimmy's bike has a speedometer which shows miles and tenths of miles. It looks like the drawing at the right.

027 8

How far will he ride before the speedometer reads

030 0

[✓ a.] about 2 miles
b. less than 2 miles
c. two-tenths of a mile
d. twenty miles

29. The number .10 means:

a. we have 10 pieces of the whole
b. it takes 10 pieces to make a whole
[✓ c.] .10 is equal to $\frac{1}{10}$
d. none of these

30. In the number 3.68:

a. six is in the hundredths column
[✓ b.] six is in the tenths column
c. six is in the one's column
d. six is equal to six hundreds

31. In the problem

$$\begin{array}{r} 3.1 \\ -\ .5 \\ \hline \end{array} :$$

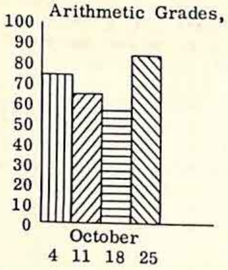
a. we cannot work problem because .5 is more than .1
[✓ b.] we change the 3 to a 2 and add 10 tenths to 1 making 11 tenths in all
c. forget the decimals and take 5 from 31
d. none of these answers are correct

32. 5.5 is equal to:

a. five and five hundredths
b. five and five tenths
c. five and one-half
[✓ d.] both b and c
e. none of these

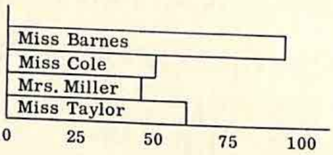
33. Mark kept a bar graph of his grades in multiplication tests. What day did he make the best grade?

a. October 4
b. October 11
c. October 18
[✓ d.] October 25



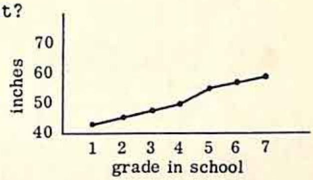
34. We are getting memberships for P.T.A. at our school. Our class has the most memberships. Who has the least?

a. Miss Barnes' room
b. Miss Cole's room
[✓ c.] Mrs. Miller's room
d. Miss Taylor's room



35. Betty's mother kept a chart of Betty's height as she entered each grade. Between what grades did she grow the most?

a. between 2nd and 3rd grade
b. between 3rd and 4th grade
[✓ c.] between 4th and 5th grade
d. between 5th and 6th grade



36. In finding the area of a checkerboard, which of these would you do?

a. count the blocks across the bottom and top
b. count the blocks up both sides, left and right
c. count the blocks across the top and down one side
[✓ d.] count all of the blocks on the board

37. In order to decide how long a piece of wrapping paper to cut to cover a gift box completely, would you measure for the:

[✓ a.] perimeter
b. area
c. length
d. width

38. For which of these must you find the perimeter in solving a problem?

a. covering a floor with linoleum
[✓ b.] making a picture frame
c. painting a floor
d. covering the top of a table

39. How many square inches are there in ½ square foot?

a. 45
b. 63
c. 54
[✓ d.] 72

40. Ann poured the fudge she had made into an 8" x 12" pan. When it was cool, she cut the fudge into 1-inch squares. How many pieces were there?

a. 45
b. 72
c. 54
[✓ d.] 96

41. To find out what time a plane is leaving for New York it would be best to look at:

a. a time gauge
b. a stop watch
c. an hour glass
d. a sun dial
[✓ e.] a timetable

42. If it was 9 a.m. in Los Angeles, and there are four time belts in the United States, how would you find out the time in New York?

a. add 4 hours
b. subtract 3 hours
[✓ c.] add 3 hours
d. change hours to minutes

43. In order to determine the number of hours when adding 4 hours, 2 hours and 25 minutes, 45 minutes, you would first:

a. add both of these
b. change hours and minutes
[✓ c.] change minutes to hours
d. change hours to minutes

44. The symbol 70° means:

a. add 70
b. square 70
[✓ c.] 70 degrees
d. 700

45. Which step needs to be taken to solve the problem shown below?

$$\begin{array}{r} 4\text{ hr. }20\text{ min. }10\text{ sec.}\\ \underline{-\ 1\text{ hr. }30\text{ min. }12\text{ sec.}}\end{array}$$

a. change seconds and minutes and subtract
b. add all elements and subtract
[✓ c.] change hours, minutes, seconds and subtract
d. change all factors to hours and subtract
e. none of these

Arithmetic Comprehension Test
Form IC Grade 6

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Instructions to students: This test will show how well you understand arithmetic. Answer all items. Usually one best answer is asked for. However, a few questions ask for the selection of several correct answers.

DO NOT MARK ON THIS TEST. Place your answers on the answer sheet.

1. How many hundreds are in 160? (a) 0 [✓ (b) 1] (c) 6 (d) 16 (e) 60

2. Which number has a 5 in the thousands place?

   (a) 3605 [✓ (b) 5523] (c) 4505 (d) 6555 (e) 7454

3. Round off 10,660 to the nearest thousand.

   (a) 10,000 (b) 10,500 (c) 10,600 [✓ (d) 11,000] (e) 10,700

4. When adding the problem shown, we think 4 and 7 are 11. We write the one and carry one. What is the number we are carrying?

$$\begin{array}{r} 267 \\ +384 \\ \hline 1 \end{array}$$

   (a) 1 (b) 10 tens (c) 11 [✓ (d) 1 ten] (e) none of these

5. Subtracting 36 from 424, we cannot subtract 6 from 4 without regrouping. (Sometimes called borrowing.) What is the number borrowed?

   [✓ (a) 1 ten] (b) 20 tens (c) 10 tens (d) 14 (e) 1

6. In the number (4,267,917,811) the numbers 267 stand for:

   a. 2 hundreds, 6 tens, 7 ones
   b. 2 thousands, 6 hundreds, 7 tens
   [✓ c. 2 hundred million, 6 ten millions, 7 millions]
   d. 2 billion, 60 millions, 7 thousands
   e. none of these

7. A farm boy milks two cows. The daily amount of milk obtained is 5 gallons and 1 quart from one cow and 4 gallons and 1 quart from the other. How many pints can be filled from each day's total milk supply?

   (a) 39 pints (b) 22 pints [✓ (c) 76 pints] (d) 156 pints (e) 78 pints

8. In the problem shown at the right, the number 1284 is best explained as being the product of:

$$\begin{array}{r} 214 \\ \times 365 \\ \hline 1070 \\ \searrow 1284 \\ 642 \\ \hline 78,110 \end{array}$$

   a. 600 X 214
   [✓ b. 60 X 214]
   c. 6 X 214
   d. 6 X 214 without a zero added
   e. none of these

9. What does the remainder mean?

$$\begin{array}{r} 5 \\ 4\overline{)22} \\ \underline{20} \\ \searrow 2 \end{array}$$

a. two-fourths; two of the units of the dividend left over.
b. two-fourths; two of the 4 units of the divisor.
[✓ c.] two; two of the 22 units left over after subtracting 4 from 22 five times.
d. two; it is the fractional part of the dividend.

10. To multiply six by fifteen is to:

a. increase six by fifteen
b. add six fifteens
c. increase six by ten times fifteen
[✓ d.] add fifteen sixes

11. In solving the subtraction problem, the best answer would be:

$$\begin{array}{r} .265 \\ \underline{-.256} \end{array}$$

a. .109
b. .019
c. .09
[✓ d.] .009
e. none of these

12. What does the circled zero mean?

↘ 1⓪.000

a. ten thousands
b. no ten thousands
[✓ c.] no thousands
d. no ten hundreds

13. Using air transportation, you can leave Los Angeles at 10:00 A.M. (PST) and arrive in New York City at 10:00 P.M. (EST). How many hours would you be in the airplane? REMEMBER TO USE THE TIME ZONES.

a. 12 hours
b. 8 hours
c. 10 hours
[✓ d.] 9 hours
e. 15 hours

14. Explain WHY moving the decimal point does not change the answer:

$.5\overline{)35.55}$ or $\underset{\curvearrowright}{5.}\overline{)35.\underset{\curvearrowright}{\dot{5}.}}$

a. dividing by a decimal you move the decimal point in the divisor and dividend
b. you cannot divide by a decimal without moving the decimal
c. when you move the decimal to the right, you have to move it the same number of places in the dividend
[✓ d.] you really multiply the divisor and the dividend by the same number and do not change relationships

15. Division may be regarded as repeated subtraction with the quotient counting the number of times the divisor can be subtracted from the dividend. With this approach to division, WHY is the 3 in the quotient placed above the 4 in the dividend?

$$\begin{array}{r} \rightarrow 3 \\ 4\overline{)144} \\ \underline{12} \\ 2 \end{array}$$

a. 4 is the last number used in the dividend
b. the 4 indicates hundreds
[✓ c.] the quotient number must be in tens
d. the quotient number must be in ones
e. answer is not given

16. Which example has a 5 in the hundredths place?

(a) .0450 [✓ (b)] .0513 (c) .5325 (d).0045

17. Which number is less than 2.02?

(a) 2.035 (b) 2.1 (c) 2.039 [✓ (d)] 2.014

18. Which number is larger than 2.04?

(a) 2.035 (b) 2.039 [✓ (c)] 2.1 (d) 2.014

19. The base of our number system is:

(a) per cent (b) zero (c) units [✓ (d)] ten (e) one

20. What number is the same as ten and one tenth?

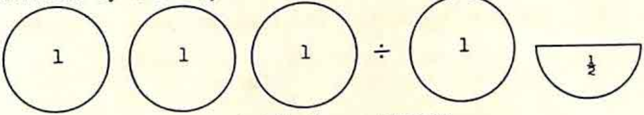
(a) 100.10 (b) 1.10 [✓ (c)] 10.10 (d) 10.01

21. To understand multiplication of fractions one must see that $\frac{1}{2}$ X 8 is:

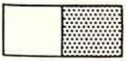
a. 16
[✓ b.] one-half as much as 1 X 8
c. both a and b
d. $4\frac{1}{2}$
e. none of these

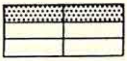
22. Show the division by marking the circles on the answer sheet.

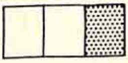
$3 \div 1\frac{1}{2}$

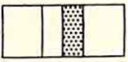


23. Which shaded figure shows one-half of one-third?

(a) 

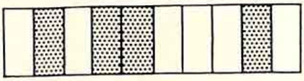
(b) 

(c) 

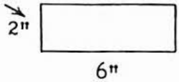
[✓ (d)] 

24. What part of the rectangle is shaded?

[✓ (a)] 0.4 (b) 4.0 (c) 0.6 (d) .04



25. To find the perimeter of the rectangle shown:



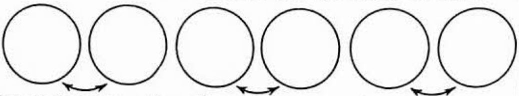
a. multiply 2" X 6"
b. divide 6" X 2"
[✓ c.] add both sides and both ends
d. none of these

26. Which of the following are uses of zero?

a. a placeholder
b. a point of origin
c. to show not any
[✓ d.] a, b, c
e. a, b

27. What proportion of the total number of circles is in each group?

(a) $\frac{2}{3}$ [✓ (b)] $\frac{2}{6}$ (c) $\frac{6}{2}$ (d) $\frac{3}{2}$

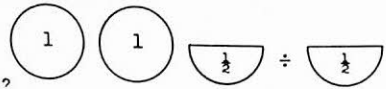


28. When the numerator and denominator of a fraction are the same, the value of the fraction is:

a. greater than 1
b. less than 1
[✓ c.] equal to 1
d. either 1 or 2
e. none of these

29. Show how the division is made for the problem below by marking the circles or shading the circles on the answer sheet.

$2\frac{1}{2} \div \frac{1}{2}$



30. How many pounds are in a ton?

(a) 1,000 [✓ (b)] 2,000 (c) 3,000 (d) 100 (e) 1,500

31. If you had two gross of pencils, how many dozen would you have?

(a) 16½ [✓ (b)] 24 (c) 12 (d) 8½ (e) none of these

32. As the numerator of a fraction decreases and the denominator remains the same, the value of the fraction:

[✓ a.] decreases
b. increases
c. remains same
d. approaches one

33. A figure with an area of 8 square inches means:

[✓ a.] 8 squares of one square inch could be put inside it
b. the square could be cut into a strip 8 inches long
c. the sum of the length of the sides is 8 inches
d. it would take 64 cubes to fill up the square

34. In the number 425 the number 2 is in:

a. one's place
b. thousand's place
[✓ c.] ten's place
d. hundred's place

35. In adding the problem $\begin{array}{r}6\\9\\\underline{6}\end{array}$ the number 15 is:

a. the sum
b. the answer
c. the product
[✓ d.] the unseen number (we think)

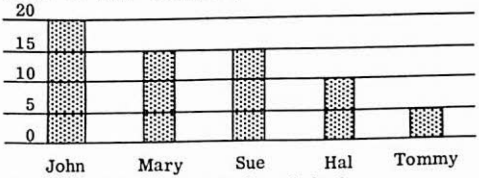
36. Twenty-seven and six more means: $\begin{array}{r}27\\\underline{+\ 6}\end{array}$

a. add 13 and 2
[✓ b.] add 13 and 20
c. add 13 and 8
d. add 42 and 12

37. A peanut butter sandwich weighs about:

a. 3 lbs.
[✓ b.] 4 oz.
c. 2 T.
d. 2 qts.

38. Look at the example:



Who got the most words right?

[✓ a.] John
b. Mary
c. Hal
d. Tommy

39. How many words did Mary get right?

a. 10
b. 20
[✓ c.] 15
d. 5

40. The picture shown in number 38 is called:

a. a word picture
b. a spelling test
[✓ c.] a bar graph
d. an arithmetic picture

41. Look at the problem $\begin{array}{r} 82 \\ \underline{-15} \end{array}$. To subtract this problem, we must:

[✓ a.] make the 2 ones larger
b. make the 1 ten larger
c. make the 5 ones larger
d. none of these

42. Tell what the answer means in the problem shown. $\begin{array}{r} 9\ \text{rl} \\ 5\overline{)46} \\ \underline{45} \\ 1 \end{array}$

a. 1 left over
b. did not finish dividing
[✓ c.] 9 groups of 5 and 1 left over
d. add 5,9,1

43. To check a division problem:

a. divide the quotient by the number divided
b. multiply the quotient by the number divided
[✓ c.] multiply the quotient by the divisor
d. divide the quotient by the divisor

44. In the problem $\begin{array}{r} 465 \\ \underline{-123} \\ 2 \end{array}$ we are subtracting:

a. 2 ones from 6 tens
[✓ b.] 2 tens from 6 tens
c. 2 hundreds from 6 hundreds
d. 2 ones from 6 ones

45. In the problem $\begin{array}{r} 27 \\ \underline{+6} \end{array}$ we think:

a. 6 ones and 7 ones are 13 tens, carry three ones
b. 6 ones and 7 ones are 13 ones, write 13 in one's place
c. 6 ones and 7 ones are 13 ones, carry 1 one
[✓ d.] 6 ones and 7 ones are 13 ones, carry 1 ten

46. In the problem $\begin{array}{r} 480 \\ \underline{\times\ \ 5} \end{array}$ we think:

a. 5 X 8 tens are 40 tens, carry 0
b. 5 X 8 tens are 40 ones, carry 4 tens
c. 5 X 8 tens are 40 hundreds, carry 4 hundreds
[✓ d.] 5 X 8 tens are 40 tens, carry 4 hundreds

47. In the problem $\begin{array}{r} 2000 \\ \underline{-1465} \end{array}$ the first thing to do is:

a. change the sign
b. write zero in each place you cannot subtract
[✓ c.] regroup each place in 2000
d. add in the one's, ten's, and hundred's place

48. In multiplying the problem $\begin{array}{r} \$0.01 \\ \underline{\times\ \ 8} \end{array}$ the answer would be written:

a. .8
b. $.8
[✓ c.] $.08
d. $8.00

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Careful research and review of literature and definition of meaningful teaching of arithmetic make the book especially helpful to the prospective teacher.

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